University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C63F 2017/18

Midterm Exam

Monday, October 30, 2017, 5:10–7:10 pm

20pt

FOR MARKERS ONLY	
Question	Marks
1	/ 20
2	/ 30
3	/ 15
4	/ 20
5	/ 15
TOTAL	**/100**

No books or calculators may be used

You may use any theorems stated in class, as long as you state them clearly and correctly.

1.

(1) (20 pts) Consider two parametrizations of the two-sphere,

(a). Spherical polar coordinates;

$$\mathbf{x}(u,v) = \big(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\big)$$

(b) Stereographic projection:

$$\mathbf{x}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

(i) Which one of these parametrizations is conformal (both, only one or neither)?

(ii) If only one is conformal, specify which one it is. Supply a proof.

Recall that a parametrization is conformal iff $E = G$ and $F = 0$

2.

(2) (30 pts) Find the Frenet apparatus $(\vec{T}, \vec{N}, \vec{B}, \kappa, \tau)$ for the arclength-parametrized curve

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right)$$

Note that the Frenet equations state that

$$\vec{T}' = \kappa\vec{N}$$
$$\vec{N}' = -\kappa\vec{T} + \tau\vec{N}$$

$$\vec{B}' = -\tau\vec{N}$$

3.

(3) (15pts)

The isoperimetric inequality states that

$$L^2 \geq 4\pi A$$

where L is the length of a simple closed plane curve and A is the enclosed area.

(a) Is it possible to have a square of length 4π and area 4π?

(b) Find a simple closed curve with length 4π and area 4π.

4.

(4) (20 pts) Find the second fundamental form for the torus

$$x(u,v) = ((a + b\cos(u))\cos(v), (a + b\cos(u))\sin(v), b\sin(u))$$

Here the second fundamental form is

$$II = \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

where

$$\ell = \mathbf{x}_{uu} \cdot \hat{n}$$
$$m = \mathbf{x}_{uv} \cdot \hat{n}$$
$$n = \mathbf{x}_{vv} \cdot \hat{n}$$

in terms of the surface normal $\hat{n}$.

5.

(5) 15pts

(6) (20 pts) Find the principal curvatures and principal directions for the torus. (These are the eigenvalues and eigenvectors of the shape operator

$$S = I^{-1} \circ II.$$

The torus is as in question 4.