UNIVERSITY OF TORONTO AT SCARBOROUGH
DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES
TERM EXAMINATION
MATC34H3 COMPLEX VARIABLES I

Examiners: L. Zhao Date: Monday 24 October 2005
Duration: Two(2) Hours, 17:00 - 19:00

I. (12 Points) Express the following in the form $a + bi$ with $a, b \in \mathbb{R}$.

i. $(1+i)^{25}$.

$1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$. Hence we have

$$(1+i)^{25} = 2^{12}\sqrt{2}(\cos 25\pi/4 + i \sin 25\pi/4) = 2^{12}\sqrt{2}(\cos \pi/4 + i \sin \pi/4) = 2^{12} + i2^{12}.$$

ii. $\dfrac{1 + i \tan\theta}{1 - i \tan\theta}$, with $\theta \in \mathbb{R}$.

Upon multiplying top and bottom of the above expression by $1 + i \tan\theta$ and noting that $1 + \tan^2\theta = \sec^2\theta$, we have

$$\frac{1 + i \tan\theta}{1 - i \tan\theta} = \frac{1 - \tan^2\theta + 2i \tan\theta}{\sec^2\theta} = (\cos^2\theta - \sin^2\theta) + 2i\frac{\tan\theta}{\sec^2\theta}.$$

iii. $e^{\pi i/6}$.

$$e^{\pi i/6} = \cos\frac{\pi}{6} + i \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

II. (5 Points) Find the two values of $\sqrt{-5 - 12i}$.

If $(x + iy)^2 = -5 - 12i$, then we have

$$x^2 - y^2 = -5 \text{ and } 2xy = -12.$$

These give

$$4x^4 + 20x^2 - 144 = 0 \text{ and } y = -6/x.$$

We must have

$$x = \pm\sqrt{\frac{-5 + \sqrt{(-5)^2 + (-12)^2}}{2}} = \pm 2,$$

and hence

$$y = -6/x = \mp 3.$$

Therefore the two square roots of $-5 - 12i$ are $2 - 3i$ and $-2 + 3i$.

III. (16 Points) Evaluate the integral

$$\int_{|z-2i|=R} \frac{z^2+3}{z(z^2+1)} \mathrm{d}z,$$

for the following values of R.

We first rewrite the integrand using partial fraction decomposition. We have

$$\frac{z^2+3}{z(z^2+1)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}$$

from which we infer that

$$Az^2 + A + Bz^2 - Biz + Cz^2 + Ciz = z^2 + 3$$

and hence

$$A = 3,\ -Bi + Ci = 0,\ A + B + C = 1 \text{ or } A = 3,\ B = C = -1.$$

Now we have

$$\frac{z^2+3}{z(z^2+1)} = \frac{3}{z} + \frac{-1}{z+i} + \frac{-1}{z-i}.$$

i. $R = \frac{1}{2}$.

The integral gives zero because the circle encloses none of the poles.

ii. $R = \frac{3}{2}$.

The circle encloses only the pole at $z = i$, hence the integral is $-2\pi i$.

iii. $R = \frac{5}{2}$.

The circle encloses the poles at $z = i$ and $z = 0$, hence the integral is $-2\pi i + 3 \times 2\pi i = 4\pi i$.

iv. $R = \frac{7}{2}$.

Now the circle encloses all three poles and hence the integral is $4\pi i - 2\pi i = 2\pi i$.

IV. (15 Points) Write down the Laurent series expansion of the following functions at the points z_0's given. Determine the domain where the series converges.

i. $f(z) = \frac{z}{(z+2)^2}$, $z_0 = -2$.

$$\frac{z}{(z+2)^2} = \frac{1}{(z+2)^2}(-2 + (z+2)) = -\frac{2}{(z+2)^2} + \frac{1}{z+2}.$$

This is actually a finite sum, hence converges wherever the $f(z)$ is defined.

ii. $g(z) = \frac{1}{z^3(z-1)}$, $z_0 = 1$.

$$\frac{1}{z^3(z-1)} = \frac{1}{2}\frac{1}{z-1}\left(\frac{1}{z}\right)'' = \frac{1}{2}\frac{1}{z-1}\left(\frac{1}{1+(z-1)}\right)'' = \frac{1}{2}\frac{1}{z-1}\left(\sum_{k=0}^{\infty}(-1)^k(z-1)^k\right)''$$

$$= \frac{1}{2}\frac{1}{z-1}\left(\sum_{k=0}^{\infty}(-1)^{k+2}(k+1)(k+2)(z-1)^k\right) = \sum_{k=0}^{\infty}\frac{1}{2}(-1)^k(k+1)(k+2)(z-1)^{k-1}.$$

The above computation is valid if $0 < |z-1| < 1$. Hence that is the domain of convergence.

iii. $h(z) = \frac{e^z}{z-3}$, $z_0 = 3$.

$$\frac{e^z}{z-3} = e^3\frac{e^{z-3}}{z-3} = \frac{e^3}{z-3}\sum_{n=0}^{\infty}\frac{(z-3)^n}{n!} = \sum_{n=0}^{\infty}\frac{e^3}{n!}(z-3)^{n-1}.$$

The above computation is valid everywhere $h(z)$ is defined. Hence the domain of convergence is $\mathbb{C} - \{3\}$.

V. (12 Points) Find all z's satisfying the following equations.

i. $z^7 = 1$.

The solutions here are the seventh root of unity. They are

$$\exp\left(\frac{k}{7}2\pi i\right), \quad k = 0, 1, \cdots, 6.$$

ii. $\exp(3z) = 2$.

There is no ambiguity in the meaning of $\log x$ if $x \in \mathbb{R}$. Hence we have

$$3z = \log 2 + 2\pi k i, \text{ for any } k \in \mathbb{Z}.$$

Therefore, the solutions that we need are

$$z = \frac{1}{3}\log 2 + \frac{2}{3}\pi k i, \text{ for any } k \in \mathbb{Z}.$$

iii. $3z^2 + \sqrt{5}iz + i = 0$.

Completing the square, we get

$$\left(z + \frac{\sqrt{5}}{6}i\right)^2 = \frac{-5 - 12i}{36}.$$

Using the answer of problem II of this exam, we have

$$z + \frac{\sqrt{5}}{6}i = \pm\frac{1}{3} \mp \frac{1}{2}i.$$

Therefore, the solutions we are looking for are

$$z = \pm\frac{1}{3} - i\frac{\sqrt{5} \pm 3}{6}.$$

VI. (10 Points) Determine the entire function $f = u + iv$ satisfying

$$f(0) = i \text{ and } u(x, y) = 2x^3y - 2xy^3 + x^2 - y^2.$$

Since f is entire, we must have that u and v satisfy the Cauchy-Riemann equations. It must be true that

$$v_x(x, y) = -u_y(x, y) = -2x^3 + 6xy^2 + 2y, \text{ and } v_y(x, y) = u_x(x, y) = 6x^2y - 2y^3 + 2x.$$

The first of the above questions only holds if

$$v(x, y) = -\frac{1}{2}x^4 + 3x^2y^2 + 2xy + g(y),$$

where $g(y)$ is a function of the variable y alone. Therefore, we have

$$v_y(x, y) = 6xy + 2x + \frac{\mathrm{d}}{\mathrm{d}y}g(y).$$

If this is to be the same as $6x^2y - 2y^3 + 2x$, we have have

$$\frac{\mathrm{d}}{\mathrm{d}y}g(y) = -2y^3$$

for all y from which we infer that

$$g(y) = -\frac{1}{2}y^4 + c$$

for some constant c and that

$$v(x,y) = -\frac{1}{2}x^4 + 3x^2y^2 - \frac{1}{2}y^4 + 2xy + c.$$

Since $i = f(0) = u(0,0) + iv(0,0)$, we must have that $c = 1$. Therefore,

$$f(x,y) = 2x^3y - 2xy + x^2 - y^2 + i\left(-\frac{1}{2}x^4 + 3x^2y^2 - \frac{1}{2}y^4 + 2xy + 1\right).$$

VII. (10 Points) Find the first four terms in the power series expansion at $z = 0$ of

$$\exp\left(\frac{1}{1-z}\right).$$

Denote the above function as $f(z)$. The first, second and third derivative of the above function, respectively, are

$$f'(z) = \exp\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^2},$$

$$f''(z) = \exp\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^4} + \exp\left(\frac{1}{1-z}\right)\frac{2}{(1-z)^3}, \text{ and}$$

$$f'''(z) = \exp\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^6} + \exp\left(\frac{1}{1-z}\right)\frac{6}{(1-z)^5} + \exp\left(\frac{1}{1-z}\right)\frac{6}{(1-z)^4}.$$

Evaluating these derivatives at $z = 0$, we have, respectively,

$$e, 3e, \text{ and } 13e.$$

Therefore, the first four terms of the power series expansion of the function in question is

$$e, ez, \frac{3}{2}ez^2, \text{ and } \frac{13}{6}ez^3.$$

VIII. (10 Points) Suppose $z_1, z_2 \in \mathbb{C}$. Prove the identity

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

We know that $|z|^2 = z\bar{z}$. Then the left-hand side of the have is

$$\begin{aligned}(z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)} &= (z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2)(\overline{z_1}-\overline{z_2}) \\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\ &= 2(z_1\overline{z_1} + z_2\overline{z_2}) = 2(|z_1|^2 + |z_2|^2).\end{aligned}$$

IX. (10 Points) Verify that the Cauchy-Riemann equations take the following form in polar coordinates

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},$$

where $x = r\cos\theta$, $y = r\sin\theta$.

We first note that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta, \tag{1}$$

$$\frac{\partial u}{\partial \theta} = r\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right), \tag{2}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta, \tag{3}$$

(4)
$$\frac{\partial v}{\partial \theta} = r\left(-\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right),$$

If we have

(5)
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then (1) becomes

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

Comparing the above with (4), we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Similarly, (5) gives that (2) is

$$\frac{\partial u}{\partial \theta} = r\left(-\frac{\partial v}{\partial y}\sin\theta - \frac{\partial v}{\partial x}\cos\theta\right).$$

When compared with (3), we infer from the above that

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Conversely, if we have

(6)
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},$$

then we infer from (1), (2), (3) and (4) that

$$\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$

and

$$-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta = -\frac{\partial v}{\partial x}\cos\theta - \frac{\partial v}{\partial y}\sin\theta.$$

These give the system of homogeneous linear equations

$$\cos\theta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \sin\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \text{ and } -\sin\theta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \cos\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

with indeterminants

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

The determinant of the coefficient matrix is $\cos^2\theta + \sin^2\theta = 1$ and in particular not zero, hence the above system has only the trivial solution. We have

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

which are the Cauchy-Riemann equations in rectangular coordinates.