MATC34 – Fall 2006

MIDTERM EXAM

November 4, 2006, 9-11 AM

1. Find all solutions of the following equation:

$$z^6 + (i-1)z^3 - i = 0$$

Let $w = z^3$. Then $w^2 + (i-1)w - i = 0$ which gives the solutions $w_1 = 1$ and $w_2 = -i$. We can see this by factorizing the polynomial, or we could use the usual formula:

$$w_{1,2} = \frac{1 - i \pm \sqrt{(i-1)^2 + 4i}}{2} = \frac{1 - i \pm \sqrt{(i+1)^2}}{2} = \frac{1 - i \pm (i+1)}{2}$$

Now we need to solve $z^3 = 1$ and $z^3 = -i$. The first gives the third roots of unity as solutions: $z_1 = 1, z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}, z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$. To solve the second one we write $i = e^{i\frac{3}{2}\pi}$. From $z^3 = e^{i\frac{3}{2}\pi}$ we get $z_4 = e^{i\frac{1}{2}\pi} = i, z_5 = e^{i(\frac{1}{2}+\frac{2}{3})\pi} = -\frac{\sqrt{3}}{2} - i\frac{1}{2}, z_6 = e^{i(1/2+4/3)\pi} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$.

2. (a) Let $f : \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$. Give the definition of the following statements

   - f is differentiable at z_0

     The limit of $\frac{f(z_0+h)-f(z_0)}{h}$ exists as $h \to 0$ (along every possible sequence converging to 0 in $\mathbb{C}$).

   - f is analytic at z_0

     f is differentiable in a neighborhood of z_0.

   (b) Suppose f is an entire function with $f(x + i\,y) = u(x) + i\,v(y)$ where u, v are real functions. Prove, that $f(z) = az + b$ where a, b are constants.

   By the Cauchy-Riemann equations, if f is entire and $f(x + i\,y) = u(x, y) + iv(x + iy)$ then $u_x = v_y$ and $u_y = -v_x$. The first equation in our case tells us that $u'(x) = v'(y)$. The left-hand side depends on x, the right-hand side depends on y, so the only way this could work if $u'(x) = v'(y) = a$ where a is a (real) constant. This means that $u(x) = a\,x + b_1$ and $v(y) = a\,y + b_2$ and thus $f(z) = a\,z + b_1 + ib_2$.

3. Give an example where the value of the line integral depends on the path of integration. I.e. find smooth curves C_1, C_2 with the same starting and end points and a function f which is analytic on both curves for which

$$\int_{C_1} f \neq \int_{C_2} f.$$

If we integrate f on C_1 and then on $-C_2$ then we get a closed path. If the two integrals don't give the same value, then the integral on this closed curve is not 0, so f cannot be entire. One possibility: let $f(z) = 1/z$, C_1 is the upper half of the $|z| = 1$ circle (i.e. $z(t) = e^{i\,t}$, $0 \leq t \leq \pi$) and C_2 is the lower half of the same circle ($e^{-i\,t}$, $0 \leq t \leq \pi$). Since the integral of $1/z$ on the whole circle is $2\pi i$, the two integrals cannot be the same. (Actually, one is πi, the other is πi.)

4. Suppose that f is an entire function and $f(0), f'(0), f^{(2)}(0), f^{(3)}(0), \ldots$ are all real. Prove that

$$f(\overline{z}) = \overline{f(z)}$$

f is entire, so it has a power series representation around 0: $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We know that $a_k = \frac{f^{(k)}(0)}{k!}$, so all coefficients are real. Then (using $(\overline{z})^k = \overline{z^k}$)

$$\overline{f(z)} = \overline{\sum_{k=0}^{\infty} a_k z^k} = \sum_{k=0}^{\infty} a_k \overline{z}^k = f(\overline{z})$$

5. Show that for all $z \in \mathbb{C}$

$$\sin(\pi/2 - z) = \cos(z).$$

Both sides define entire functions, and for $z \in \mathbb{R}$ the two sides are equal. Thus by the Uniqueness Theorem the identity holds for all $z \in C$.
Or: use the definitions of sin and cos using the exponential function and prove the identity directly.

6. (a) State the Cauchy Integral Formula for entire functions.

   Suppose f is entire, C is a circle and a is inside the circle. Then $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$, where the path of integration is in the positive direction.

   (b) Explain how it can be used to show that an entire function has a power series representation. (Give only the main idea of the proof, don't worry about the details.)

   We applied the Cauchy Integral Formula for a fixed z and a large enough circle with the origin as its center. We expressed $\frac{1}{w-z}$ as a power series in z (using $|w| > |z|$) and then exchanged the integrals and the summation (we can do that since the convergence is uniform). This gives the power series expansion around 0, and from that we can get the general case by shifting the whole picture.

7. Suppose that f is analytic and bounded by 1 on the disk $|z| \leq 2$ and that $f(1/2) = 0$. Show that

$$|f(1)| \leq \frac{2}{7}$$

Let $g(z) = f(2z)$. Then g is analytic and bounded by 1 on the disk $|z| \leq 1$ and $g(1/4) = 0$. By the usual arguments we get that $|g(1/2)| \leq |B_{1/4}(1/2)| = \frac{1/2-1/4}{1-1/8} = 2/7$. Since $g(1/2) = f(1)$, this proves the inequality.

8. Suppose that f is an entire function for which

$$|f(z)| \leq |\sin(z)| \qquad \text{and} \qquad |f(z)| \leq |\cos(z)|.$$

Show that $f(z) = 0$ for all $z \in \mathbb{C}$.
(Hint: First find all the entire functions satisfying the first inequality.)

If $|f(z) \leq |\sin(z)|$ then $g(z) = \frac{f(z)}{\sin(z)}$ must be an entire function (if $\sin(z) = 0$ then $f(z)$ must be 0 as well, so we can define $g(z)$ analytically there) and $|g(z)| \leq 1$. But by the Liouville Theorem $g(z)$ must be constant (with modulus at most one), so $f(z) = c_1 \sin(z)$ with $|c_1| \leq 1$. Similarly, if we look at the second inequality, we get that $f(z)$ must be equal to $c_2 \cos(z)$ with $|c_2| \leq 1$. Thus $f(z) = c_1 \sin(z) = c_2 \cos(z)$ for all $z \in \mathbb{C}$, and this can only hold if $c_1 = c_2 = 0$.