University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2013/14

Midterm Exam: Solutions

Friday, November 1, 2013; 120 minutes

No books or calculators may be used

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(25 points)** Is the function $f(z)$ defined by

$$f(z) = z^2\bar{z}$$

differentiable at $z = 0$? If you think so, give a proof and compute $\frac{df}{dz}$ at this value; if you think not, show why the complex derivative at 0 does not exist.

Solution:

$$f'(z) = \lim_{h\to 0} \frac{(h+z)^2\overline{h+z} - z^2\bar{z}}{h}$$

$$= \lim_{h\to 0} \frac{z^2\bar{h} + 2z\bar{z}h}{h}$$

$$= z\bar{z} + \lim_{h\to 0} z^2\frac{\bar{h}}{h}$$

This limit only exists when $z = 0$ (because in polar coords $\bar{h}/h = e^{-2i\theta}$ and the limit depends on θ, so it makes sense only when $z = 0$ so that the function of h of which we are taking the limit is 0)

2. **(25 points)** Let γ denote the contour around the boundary of the unit disc $|z| \le 1$, oriented counterclockwise. Evaluate the following integrals:

(a) $\int_\gamma \frac{1}{(z-3)(z-4)}dz$

Soln: (a)The function is non-holomorphic only at 3 and 4 and these points are outside γ. Hence by Cauchy the integral is 0

(b) $\int_\gamma z|z|^4 dz$

Soln: Restricted to the unit circle, this function is z, which is holomorphic. So the integral is 0 by Cauchy

(c) $\int_\gamma \frac{1}{z^2} dz$

Soln: The line integral is

$$\int_0^{2\pi} e^{-2it} \cdot (ie^{it}) dt = \int_0^{2\pi} ie^{-it} dt = 0$$

3. **(25 points)** Let f be the function

$$f(z) = \frac{1}{4 + z^2}$$

on the unit disk $\{z \in \mathbf{C} : |\mathbf{z}| \leq \mathbf{1}\}$.

(a) What is the maximum value of $|f(z)|$ on the disk?

Solution: The Maximum modulus theorem says the maximum of the absolute value is attained on the boundary. So restrict to $z = e^{i\theta}$, and look for a minimum in $|4 + e^{2i\theta}|^2 = 17 + 8\cos 2\theta$. The minimum happens when $\theta = \pm\pi/2$ ($z = \pm i$), and the value of f is $1/3$.

(b) At what value(s) of z is the maximum value of $|f(z)|$ attained? Answer: The value of z where $|f|$ takes its maximum value is $1/3$.

State all theorems you use.

4. **(25 points)**

(a) Compute the Laurent series at $z = 1$ for the following function:

$$f(z) = \frac{1}{z^2 - 1}$$

Answer:

$$\begin{aligned} \frac{1}{z^2 - 1} &= \frac{1}{(z-1)(z-1+2)} \\ &= (z-1)^{-1} \frac{1}{2} \frac{1}{1 + (z-1)/2} \end{aligned}$$

Now use

$$(1 + w)^{-1} = \sum_{n=0}^{\infty} (-1)^n w^n$$

with $w = (z-1)/2$ The radius of convergence of this series is 2 in other words the series converges if $|z - 1| < 2$.

(b) Classify the singularities of the following functions, and state the orders of all zeroes and poles:

   i. $\frac{1}{(z^2+1)\sin z}$
      Answer: $z = \pm i$, $z = n\pi$, poles of order 1, no zeroes

   ii. $\frac{1}{z^3(z^2+1)}$
      Answer: $z = 0$ pole of order 3 $z = \pm i$ pole of order 1