9. Conformal Mapping

Definition 9.1. *A circline is a circle or a line in the complex plane.*

Definition 9.2. *The Riemann sphere is the complex plane* $\mathbf{C}$ *with one extra point added (the "point at infinity"). We can think of this as the union of two copies of* $\mathbf{C}$*, with one of them parametrized by* z *and the other by* w *with the convention*

$$zw = 1$$

leading to $z = 0$ *iff* $w = \infty$ *and* $w = 0$ *iff* $z = \infty$.

Definition 9.3. *(inverse points)* α *and* β *are inverse points w.r.t. the circle* $|z - a| = r$ *if*

$$(\alpha - a)(\bar{\beta} - \bar{\alpha}) = r^2.$$

α *and* β *are inverse with respect to a straight line* ℓ *if* β *is the reflection of* α *in* ℓ.

Remark 9.4. *Given a pair of points and a third point, there is exactly one circline for which the pair of points are inverse points with respect to the circline and the third point is in the circline.*

9.1. **Conformal Mapping.**

Theorem 9.5. *If* f *is holomorphic in an open set* G *and* $f'(z) \neq 0$ *for some* $z \in G$*, then* f *preserves angles between lines meeting at* z.

Definition 9.6. *A mapping* f *is* conformal *in an open set* G *if* f *is holomorphic on* G *and* $f'(z) \neq 0$ *for any* $z \in G$. f *is conformal at a point* w *if it is conformal on some small disc containing* w.

A conformal mapping preserves the sign of angles as well as their size. If $f'(z) = 0$, then f does not preserve angles at z. For example the map $f(z) = z^2$ doubles the angles at $z = 0$.

9.2. **Construction of conformal maps.** If we want a conformal map to send the upper half plane $H = |\{x + iy|y > 0\}$ onto the open unit disc $D(0; 1)$, we require that f is conformal on H and f sends the line $\{x + iy|y = 0\}$ to the circle $|z| = 1$. This is accomplished by a linear fractional transformation or Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

. We find

$$f(z) = \frac{z - i}{z + i}$$

since it is easy to see that this map sends the real axis to the unit circle.

Remark 9.7. *If f and g are conformal maps, then so is $f \circ g$.*

Theorem 9.8. *If $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ are in $\mathbf{C}$ and $z_1 \neq z_2 \neq z_3, w_1 \neq w_2 \neq w_3$. Then there is a unique Möbius transformation taking z_j to w_j for all $j = 1, 2, 3$.*

So to find a conformal map taking one circline to another, we just have to solve the equation for the map to send three specified points to three other specified points.

(A line is a special case of a circline, since all lines contain the point at infinity. For any two points in the complex plane there is only one line passing through both of them.)

The procedure to find a conformal map taking one region to another is to identify such a conformal map as the composition of a number of conformal maps using a table showing some standard conformal maps and the image of the unit circle under these maps.

9.3. **Harmonic functions.**

Definition 9.9. *A real-valued function $u(x, y)$ is* harmonic *if*

(1) *The partial derivatives of u are continuous, and*
(2)
$$u_{xx} + u_{yy} = 0$$

Theorem 9.10. *If $f = u + iv$ is holomorphic, then u is harmonic.*

9.3.1. *Harmonic conjugates.* If $u(x, y)$ is harmonic, then there is a *harmonic conjugate* v for which $u + iv$ is holomorphic. This means u and v satisfy the Cauchy-Riemann equations. To find the harmonic conjugate, we solve the Cauchy-Riemann equations for v.

Example 9.1. *Let $u(x, y) = 2xy$. Then*
$$u_x = 2y.$$
Using the Cauchy-Riemann equations,
$$v_y = 2y$$
so
$$v(x, y) = y^2 + h(x)$$
Now differentiate v with respect to x:
$$v_x = \frac{dh}{dx}$$
Now impose the Cauchy-Riemann equation:
$$(v_x =) - u_y = \frac{dh}{dx}$$

Now differentiate u with respect to y:

$$u_y = 2x$$

so we have

$$\frac{dh}{dx} = -2x$$

and $h = -x^2 + C$ *where* C *is a constant. It follow that the harmonic conjugate is*

$$v = y^2 - x^2 + C$$

Notice that $-v$ *(respectively* u*) are the real (respectively imaginary) parts of* $f(z) = z^2$*.*

9.4. **Conformal maps sending the upper half plane to the unit disk.** If f sends the upper half plane to the unit cirle, let

$$f(z) = \frac{az + b}{cz + d}$$

Because $|f(z)| = 1$ when z is a real number, we have

$$\frac{ax + b}{cx + d} = 1.$$

Taking $x = 0$ we have $|b| = |d|$. Taking $x = \infty$ we have $|a| = |c|$. We may assume $a = 1$, as long as $a \neq 0$ (divide both the numerator and denominator by a so that a is replaced by 1). So we have writing $b = re^{i\theta}$

$$f(x) = \frac{x + re^{i\theta}}{e^{i\psi}x + re^{i\phi}}$$

which is on the unit circle. This implies

$$|x + re^{i\theta}| = |x + re^{i(\phi - \psi)}|$$

This implies (taking the square of the absolute value and matching the coefficients of x and x^2) that either

$$e^{i\theta} = e^{i(\phi - \psi)}$$

or

$$e^{i\theta} = e^{-i(\phi - \psi)}$$

So our map is

$$f(x) = e^{-i\psi}\left(\frac{x + re^{i\theta}}{x + re^{i\theta}}\right)$$

or

$$f(x) = e^{-i\psi}\left(\frac{x + re^{i\theta}}{x + re^{-i\theta}}\right)$$

The first map sends the entire real axis to $e^{-i\psi}$ (this is a conformal map, but not the one we want). The second map sends

$$f(x) = e^{-i\psi}\left(\frac{x+a}{x+\bar{a}}\right) \tag{1}$$

for some real number ψ and some complex number a.

The example presented in class

$$f(z) = \frac{z-i}{z+i} \tag{2}$$

takes the upper half plane to the unit circle, but it is not the only conformal map that takes the upper half plane to the unit circle. (2) is a special case of (1) (taking $\psi = 0$ and $a = -i$). All other conformal maps taking the upper half plane to the unit circle are obtained by composing with conformal maps g taking $(1, -1, i)$ to general values in the unit circle

$$g(1) = e^{iC}$$
$$g(-1) = e^{iA}$$
$$g(i) = e^{iB}$$

This means

$$\frac{a+b}{c+d} = e^{iC}$$
$$\frac{-a+b}{-c+d} = e^{iA}$$
$$\frac{a+ib}{c+id} = e^{iB}$$

$$g(z) = \frac{az+b}{cz+d}$$

where we set $a = 1$ and solve

$$b - e^{iC}c - e^{iC}d = -1 \tag{3}$$

$$b + e^{iA}c + e^{iA}d = +1 \tag{4}$$

$$ib - e^{iB}c - ie^{iB}d = -1 \tag{5}$$

Subtracting (4) from (3)

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\det(R)} R \begin{pmatrix} -2 \\ -1+i \end{pmatrix}$$

where R is the matrix

$$R = \begin{pmatrix} -e^{iC} + ie^{iB} & e^{-iC} + e^{iA} \\ e^{iC} - ie^{iB} & -e^{iC} - e^{iA} \end{pmatrix}$$

Finally we solve for b using (3).