Complex variables

Midterm Solutions

(1) Verify that $u(x,y) = e^x(x\cos y - y\sin y)$ is harmonic and find harmonic conjugate. Express the analytic function $f = u + iv$ in terms of the complex variable $z = x + iy$.

One has

$$\partial_x u = e^x(x\cos y - y\sin y) + e^x\cos y, \quad \partial_{xx} u = e^x(x\cos y - y\sin y) + 2e^x\cos y,$$
$$\partial_y u = e^x(-x\sin y - \sin y - y\cos y), \qquad \partial_{yy} u = e^x(-x\cos y - 2\cos y + y\sin y),$$
$$\partial_{xx} u + \partial_{yy} u = e^x(x\cos y - y\sin y) + 2e^x\cos y +$$
$$e^x(-x\cos y - 2\cos y + y\sin y) = 0$$

Thus u is harmonic.

We will find now the harmonic conjugate. There are always two ways to solve such problem. One can start with $\partial_y v = \partial_x u$ and to proceed with $\partial_x v = -\partial_y u$ or vice versa to start with $\partial_x v = -\partial_y u$ and to proceed with $\partial_y v = \partial_x u$. It is enough to do it one way. We do it both ways here.

$$\partial_y v = \partial_x u = e^x(x \cos y - y \sin y) + e^x \cos y,$$
$$v = \int [e^x(x \cos y - y \sin y) + e^x \cos y] dy$$

Recall that

$$\int y \sin y dy = -y \cos y + \sin y + C$$

Hence,

$$v = e^x(x \sin y + y \cos y - \sin y) + e^x \sin y + C(x) =$$
$$e^x(x \sin y + y \cos y) + C(x)$$

Furthermore

$$\partial_x v = -\partial_y u = -e^x(-x \sin y - \sin y - y \cos y)$$

On the other hand

$$\begin{aligned} \partial_x v = \partial_x[e^x(x \sin y + y \cos y) + C(x)] = \\ e^x(x \sin y + y \cos y) + e^x \sin y + C'(x) \end{aligned}$$

Hence

$$\begin{aligned} e^x(x \sin y + y \cos y) + e^x \sin y + C'(x) = -e^x(-x \sin y - \sin y - y \cos y) \\ C' = 0, \quad C = c = 0, \\ v = e^x(x \sin y + y \cos y) \end{aligned}$$

Now we do it vice versa

$$\partial_x v = -\partial_y u = -e^x(-x\sin y - \sin y - y\cos y),$$
$$v = -\int e^x(-x\sin y - \sin y - y\cos y)dx =$$
$$(xe^x - e^x)\sin y + e^x(\sin y + y\cos y) + C(y) = e^x(x\sin y + y\cos y) + C(y)$$

since

$$\int xe^x dx = xe^x - e^x$$

Furthermore,

$$\partial_y v = \partial_x u = e^x(x\cos y - y\sin y) + e^x\cos y$$

On the other hand

$$\partial_y v = \partial_y[e^x(x\sin y + y\cos y) + C(y)] = e^x(x\cos y + \cos y - y\sin y) + C'(y)$$

Hence

$$\begin{aligned} e^x(x\cos y + \cos y - y\sin y) + C'(y) = e^x(x\cos y - y\sin y) + e^x\cos y, \\ C' = 0, \quad C = c = 0, \quad v = e^x(x\sin y + y\cos y) \end{aligned}$$

Finally,

$$\begin{aligned} f(x+iy) = u + iv = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) = \\ e^x[(x\cos y - y\sin y) + i(x\sin y + y\cos y)] = \\ e^x[x(\cos y + i\sin y) + iy(\cos y + i\sin y)] = \\ e^x[(x+iy)(\cos y + i\sin y)] = e^x(x+iy)e^{iy}] = (x+iy)e^{x+iy} = ze^z \end{aligned}$$

2 **(7 marks)**

(a) State the Cauchy theorem for simply connected domains.

Let C be a closed contour (without self intersections). Let $f(z)$ be analytic in the domain D bounded by C. Then

$$\int_C f(z)dz = 0$$

(b) Using parametrization evaluate the following integral

$$\int_C (x^2 + ixy)dz$$

where C is the squire with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ and $z = x + iy$ as usual.

One has

$$\int_C (x^2 + ixy)dz = \int_{C_1} (x^2 + ixy)dz + \int_{C_2} (x^2 + ixy)dz + \int_{C_3} (x^2 + ixy)dz + \int_{C_4} (x^2 + ixy)dz$$

where C_j are the edges of the squire, $C_1 : 1 \to i$, $C_2 : i \to -1$, $C_3 : -1 \to -i$, $C_4 : -i \to i$,

$$C_1 : z = 1 + t(i-1) = 1 - t + it, 0 \le t \le 1, x = 1 - t, y = t, dz = (-1 + i)dt,$$
$$C_2 : z = i + t(-1 - i) = -t + i - it, 0 \le t \le 1, x = -t, y = 1 - t, dz = (-1 - i)dt,$$
$$C_3 : z = -1 + t(-i + 1) = t - 1 - it, 0 \le t \le 1, x = t - 1, y = -t, dz = (1 - i)dt,$$
$$C_4 : z = -i + t(1 + i) = t + it - i, 0 \le t \le 1, x = -t, y = -t - 1, dz = (1 + i)dt$$

$$\int_{C_1}(x^2+ixy)dz=\int_0^{\;1}((1-t)^2+i(1-t)t)(-1+i)dt=$$

$$(-1+i)\int_0^{\;1}((1-t)^2+i(1-t)t)dt=(-1+i)[-\frac{1}{3}(1-t)^3|_{t=0}^{t=1}+$$

$$i\frac{1}{2}t^2|_{t=0}^{t=1}-i\frac{1}{3}t^3|_{t=0}^{t=1}]=(-1+i)[\frac{1}{3}+i\frac{1}{2}-i\frac{1}{3}]=(-1+i)[\frac{1}{3}+i\frac{1}{6}],$$

$$\int_{C_2}(x^2+ixy)dz=\int_0^{\;1}((-t)^2-i(1-t)t)(-1-i)dt=$$

$$(-1-i)[\frac{1}{3}t^3|_{t=0}^{t=1}-i\frac{1}{2}t^2|_{t=0}^{t=1}+i\frac{1}{3}t^3|_{t=0}^{t=1}]=$$

$$(-1-i)[\frac{1}{3}-i\frac{1}{2}+i\frac{1}{3}]=(-1-i)[\frac{1}{3}-i\frac{1}{6}],$$

Similarly

$$\int_{C_3} (x^2 + ixy)dz = \int_0^1 ((t-1)^2 - i(t-1)t)(1-i)dt = (1-i)[\frac{1}{3} + i\frac{1}{6}],$$
$$\int_{C_4} (x^2 + ixy)dz = \int_0^1 ((-t)^2 + i(1+t)t)(1+i)dt = (1+i)[\frac{1}{3} - i\frac{1}{6}],$$

So,

$$\int_C (x^2 + ixy)dz = (-1+i)[\frac{1}{3} + i\frac{1}{6}] + (-1-i)[\frac{1}{3} - i\frac{1}{6}] +$$
$$(1-i)[\frac{1}{3} + i\frac{1}{6}] + (1+i)[\frac{1}{3} - i\frac{1}{6}] = 0$$

(c) Can one evaluate this integral via Cauchy Theorem? Give complete explanation of your answer

One can not evaluate the integral via Cauchy Theorem. The reason is that the function $f(x+iy) = x^2 + ixy$ is not analytic. Indeed $f = u + iv$ with $u = x^2$ $v = xy$, $\partial_x u = 2x$, $\partial_y v = x \neq \partial_x u$.

3 **(20 marks)**

(a) **(5 marks)**State the Cauchy formula for analytic function and its derivatives

Let C be a closed contour (without self intersections). Let $f(z)$ be analytic in the domain D bounded by C. Then for any $z \in D$ holds

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

(b) **(15 marks)** Evaluate the integral

$$\int_C \frac{dz}{z^2(z^2+16)}$$

where $C = \{|z| = 2\}$.

The singular points of the integrant are the roots of $z^2(z^2+16) = 0$. These roots are $z_0 = 0$, $z_1 = 4i$, $z_2 = -4i$. Only z_0 is inside the contour. One can write the integral in the form of Cauchy formula for the derivatives

$$\int_C \frac{dz}{z^2(z^2+16)} = \int_C \frac{f(z)}{(z-z_0)^{k+1}} dz = \frac{2\pi i}{k!} f^{(k)}(z_0)$$

with $k = 1$, $f(z) = 1/(z^2+16)$. Thus

$$\int_C \frac{dz}{z^2(z^2+16)} = 2\pi \frac{2z}{(z^2+16)^2}|_{z=0} = 0$$

4 **(20 marks)**

(a) **(5 marks)** State the Cauchy theorem for doubly connected domains

Let C, C_1 be contours without self-intersections. Let D and D_1 be the domains bounded by C and C_1 respectively. Assume $D_1 \subset D$. The domain $D' = D \setminus D_1$ is called doubly connected. The Cauchy Theorem for D' says that if $f(z)$ is analytic in D' then

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

(b) **(5 marks)** State the Cauchy estimates for the derivatives of analytic function

(c) **(5 marks)** State the Gauss mean value theorem

Let $f(z)$ be analytic in a domain containing the disk $|z - z_0| \leq R$. Then

$$|f^{(k)}(z_0)| \leq \frac{k!}{2\pi R^k} \int_0^{2\pi} |f(z_0 + Re^{it})|dt$$

For $k = 0$ the inequality

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})| dt$$

is called the Gauss mean value theorem.

(d) **(5 marks)** State the maximum value principle

Let $f(z)$ be analytic in the domain D. If f is non-constant then $|f(z)|$ never assumes maximal value inside the domain D.