UNIVERSITY OF TORONTO SCARBOROUGH
Computer & Mathematical Sciences

MAT C34 **Solution Sketch: Midterm Exam**

1. Evaluate each of the following integrals.

a) $\int_\gamma \frac{1}{z-9}\,dz$ where γ is the unit circle, oriented counterclockwise.

Soln: $\int_\gamma \frac{1}{z-9}\,dz = 0$ since $\frac{1}{z-9}$ is holomorphic in the interior of γ.

b) $\int_\gamma e^{2z}/z^5\,dz$ where γ is the unit circle, oriented counterclockwise.

Soln: Letting $f(z) = e^{2z}$, by the Cauchy Integral Formula for derivatives,

$$\int_\gamma e^{2z}/z^5\,dz = 2\pi i f^{(4)}(0)/4! = 2\pi i\,2^4 e^0/4! = 4\pi i/3$$

c) $\int_\gamma e^{3z}\,dz$ where γ is the graph of $y = \sqrt{x}$ from $(0,0)$ to $(4,2)$.

Soln: Letting $g(z) = e^{3z}/3$, the integral, which is independent of the path, is given by $\int_\gamma e^{3z}\,dz = g(4+2i) - g(0) = \frac{e^{12+6i}-1}{3}$.

2. Suppose $f : D \to \mathbb{C}$ is holomorphic and that $\operatorname{Re} f$ is constant. Show that $f(z)$ is constant.

Soln: Write $f + u + iv$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Since u is constant, $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$. Therefore by the Cauchy-Riemann equations $\partial v/\partial x = -\partial u/\partial y = 0$ and $\partial v/\partial y = \partial u/\partial x = 0$ and so v is contant. Therefore f is constant.

3. For each of the following functions, list the points where the function is singular, and for each singularity determine the type of the singularity (removable, pole, or essential). In the case of poles, give the order of the pole. (Explain your choices, but only a brief explanation is expected.)

a) $\frac{e^{1/z}}{z-1}$

Soln: The singularties are at $z = 0$ and $z = 1$. Since $e^{1/z}$ is analytic at $z = 1$, the singularity at $z = 1$ is a pole or order 1. Since $1/(z-1)$ is analytic at $z = 0$ and the Laurent series for $e^{1/z}$ about $z = 0$ has nonzero coeffiecients of z^n for all $n < 0$, the singularity at $z = 0$ is essential.

b) $\frac{1}{(z-i)\sin z}$

Soln:

The singularties are at $z = i$ and $z = n\pi$ for every integer n. Since $\lim_{z\to n\pi} \frac{z-n\pi}{(z-i)\sin z}$ exists for all n (as can be seen using Lôpital's Rule, for example), the singularites at $z = n\pi$ are all poles of order 1, and similarly the singularity at $z = i$ is a pole of order 1.

c) $\frac{e^z - e^{-z}}{z}$

Soln: The only singularty is at $z - 0$. Since $\lim_{z\to 0} \frac{e^z - e^{-z}}{z}$ exists, the singularity at $z = 0$ is removable.

4. a) Find the Laurent series expansion of $\frac{\sin z}{z^2}$ about $z = 0$ which is valid in the domain $|z| > 0$.

Soln: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ so dividing by z^2 gives

$$\frac{\sin z}{2^2} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-1}}{(2n+1)!}$$

b) Find the Laurent series expansion of $f(z) = \frac{-1}{(z-1)(z-3)}$ about $z = 0$ in some domain which includes the point $1 + i$.

Soln: The singularities of $f(z)$ are at $z = 1$ and $z = 3$ which have distance 1 and 3 respectively from $z = 0$. Since the distance of $1 + i$ to 0 is $\sqrt{2}$ which lies between 1 and 3, the Laurent series required will be the one which converges in the annulus $A := \{z \mid 1 < |z| < 3\}$. By partial fractions

$$f(z) = \frac{1}{2(z-1)} - \frac{1}{2(z-3)} = f_2(z) + f_1(z)$$

where $f_2(z) = \frac{1}{2(3-z)}$ and $f_1(z) = \frac{1}{2(z-1)}$. Since $f_2(z)$ is analytic in $|z| < 3$ the Laurent series of f_2 within A is its Taylor series

$$f_2(z) = \frac{1}{6}\sum \frac{1}{1 - z/3} = \frac{1}{6}\sum_{n=0}^{\infty} \frac{z^n}{3^n}.$$

Since $f_1(z)$ is analytic in $|z| > 1$, letting $\zeta = 1/z$, the Laurent series of $f_1(z)$ is the Taylor series of $f_1(\zeta)$ which is

$$f_1(\zeta) = \frac{1}{2(1/\zeta - 1)} = \frac{1}{2}\frac{\zeta}{1-\zeta} = \frac{\zeta}{2}\sum_{n=0}^{\infty}\zeta^n = \frac{1}{2}\sum_{n=0}^{\infty} z^{-(n+1)}.$$

Therefore the Laurent series of $f(z)$ valid at $z = 1 + i$ is

$$f(z) = \frac{1}{6}\sum \frac{1}{1 - z/3} = \frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \frac{1}{2}\sum_{n=0}^{\infty} z^{-(n+1)}.$$

5. In each of the following, compute the residue $\operatorname{Res}_a f$ for the given function $f(z)$ at the given point a.

a) $f(z) = \frac{e^z}{z^3}$; $\quad a = 0$.

Soln.: $f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \ldots$ so $\operatorname{Res}_0 f(z) = 1/2$.

b) $f(z) = \frac{e^{\pi z}}{z^2+1}$; $\qquad a = i.$

Soln: $f(z) = \frac{e^{\pi z}}{z^2+1} = \frac{e^{\pi z}}{(z+i)(z-i)} = \frac{g(z)}{z-i}$ where $g(z) = \frac{e^{\pi z}}{z+i}$. Since $g(z)$ is holomorphic at $z = i$,

$$\text{Res}_i f(z) = g(i) = \frac{e^{\pi i}}{2i} = -\frac{1}{2i} = \frac{i}{2}.$$

6. Use the methods of complex analysis to evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx$

Soln.:

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \text{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx.$$

Let $f(z) = \frac{e^{iz}}{(z^2+1)^2}$. Since $\frac{|z|}{|z|^4+1}$ is bounded as $z \to \infty$, a theorem from class tells us that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx = 2\pi i \sum \text{Residues of } f(z) \text{ in upper half plane.}$$

The only singularity of $f(z)$ in the upper half plane is at $z = i$. Write $f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2} = \frac{g(z)}{(z-i)^2}$ where $g(z) = \frac{e^{iz}}{(z+i)^2}$. Then $\text{Res}_i\, f(z) = g'(i)$. Since $g'(z) = \frac{(z+i)^2 ie^{iz} - e^{iz}2(z+i)}{(z+i)^4}$ we get

$$g'(i) = \frac{(2i)^2 ie^{-1} - e^{-1}2(2i)}{(2i)^4} = -\frac{i}{2e}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx = 2\pi i \frac{-i}{2e} = \frac{\pi}{e}$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \text{Re}\, \frac{\pi}{e} = \frac{\pi}{e}.$$