Math C31 Term test

October 30, 2004

Your name Bálint Virág	
Student ID	

There are 6 problems on this exam. Do all problems, each is worth 20 points. Start with the easier problems. This is a closed book exam. No calculators are allowed.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

Problem 1.

(a) Define: open subset of $\mathbb{C}$

A is open if for all $z \in A$ $\exists r > 0$ s.t. $D(z, r) \subset A$.

(b) Define: a function f is analytic at a point z.

f is analytic at z if for some nbhd. B of z, for all $w \in B$ the limit $\lim_{h \to 0} \frac{f(w+h) - f(w)}{h}$ exists.

(c) Define: entire function.

f is entire if it is analytic at all $z \in \mathbb{C}$.

(d) Define the line integral

$$\int_C f(z)\, dz \qquad (*)$$

If γ is a differentiable curve that parametrizes C, and $\gamma: [a,b] \to C$

then $\int_C f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$

(e) State the closed curve theorem (the version for entire functions will do).

If C is a closed curve then $(*) = 0$ for entire fcts. f.

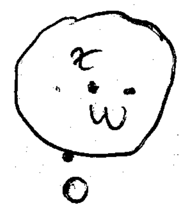
Problem 2. Let $A = \mathbb{C} \setminus \{0\}$. Prove that A is open.

Let $z \in A$, then $D(z, |z|) \subset A$,

because if $w \in D(z, |z|)$ then

* $|z - w| < |z|$, and by the

triangle inequality



$$|w| + |z - w| \geq |z|$$

$$|w| \geq |z| - |z - w| \underset{*}{>} 0.$$

So $|w| \neq 0$, $w \neq 0$ and $w \in A$.

Problem 3. Let $u : \mathbb{R} \to \mathbb{R}$ be a function with continuous derivative.
(a) Prove that $f(z) := u(\mathrm{Re}(z)) + iu(\mathrm{Im}(z))$ has a complex derivative at z if $z = (1+i)r$, where r is a real number.

(1) f has a cts. derivatice in x, y

and, setting $x = y = r$:

(2) The Caudy- Ricmann equations are satisfied: Let $\tilde{u}(x,y) = u(x)$

and $\tilde{v}(x,y) = u(y)$

then $\partial_x \tilde{u} = u'(x) = u'(r) = u'(y) = \partial_y \tilde{v}$

and $\partial_y \tilde{u} = 0 = -\partial_x \tilde{v}$

(b) Prove that if $u(x) = x^2$ then f is not an entire function. You may use any theorems we learned, but state clearly what you use.

$f(z) = x^2 + iy^2$

Check at $z = 1$ so $x = 1$, $y = 0$.

$\partial_x u = 2x = 1$

$\partial_y v = 2y = 0$

The C-R equations fail, so f is not entir.

Problem 4.

Find the **derivative** and **radius of convergence** of the following.

(a)

$$f(z) = \sum_{n=0}^{\infty} nz^n$$

$$f'(z) = \sum_{n=1}^{\infty} n^2 z^{n-1}$$

$\lim a_n^{1/n} = \lim n^{1/n} = 1$

so $\limsup |a_n|^{1/n} = 1$

$\Rightarrow R = 1.$

(b)

$$f(z) = \sum_{n=0}^{\infty} z^{(n^2)}$$

$$f'(z) = \sum_{n=1}^{\infty} n^2 z^{n^2-1}$$

$$\sup_{k \geq n} |a_k|^{1/k} = \sup_{k \geq 1} 1^{1/k} = 1$$

$$\limsup_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} 1 = 1.$$

So $R = 1.$

Problem 5.

(a) Evaluate the line integral of $f(z) = e^{iz}$ on the line segment connecting $-i$ and i. You may use any theorems that we learned, but state clearly what you are using.

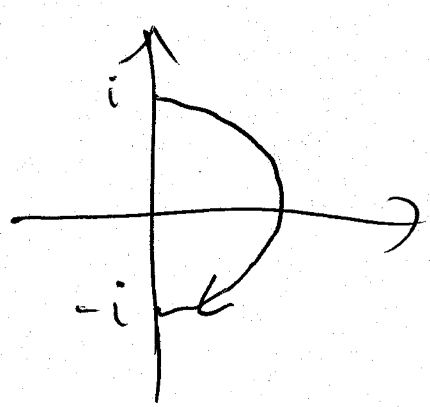
f is the derivative of $F = -i e^{iz}$

In this case, the line integral of f is just the difference of F at the endpoints (by Fundamental Thm of Calculus) (**)

$$\int_{[-i,i]} f\,dz = F(i) - F(-i) = -i(e^{-1} - e^{1})$$

(b) Evaluate the line integral of of $f(z) = e^{iz}$ clockwise on the intersection of the circle of radius 1 and the right half-plane $\{z : Re(z) \geq 0\}$. You may use any theorems that we learned, but state clearly what you are using.

By the same argument as above, we have



$$\int_C f\,dz = F(-i) - F(i)$$
$$= i(e^{-1} - e^{1}).$$

Problem 6.
Let f be an entire function, and assume that the second derivative satisfies $|f''(z)| < 1$ for all $z \in \mathbb{C}$. Show that for some constants $a, b, c \in \mathbb{C}$ we have $f(z) = az^2 + bz + c$ for all $z \in \mathbb{C}$.

Liouville's theorem says that a bdd. entire function is constant. If f is entire, then so is f'', hence $f'' = \text{const}$, say $2a$.

Let $b = f'(0)$, then by (**, last page):

$$f'(z) = \int_{[0,z]} f''(w)\,dw + f'(0)$$

$[0,z]$ ← line segment

$$= \int_{[0,z]} 2a\,dw + b$$

$$= 2az + b.$$

Similarly, let $c = f(0)$, then

$$f(z) = \int_{[0,z]} f'(z)\,dz + c = az^2 + bz + c \quad \square.$$