MATC34 2018 Solutions to Assignment 1

1. Prove that the function $\exp(\bar{z})$ is not holomorphic anywhere.

   Solution: $f(z) = \exp \bar{z}$

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\exp(\bar{z} + \bar{h}) - \exp \bar{z})}{h}$$

$$= \lim_{h\to 0} \frac{\bar{h}}{h \exp \bar{z}}$$

   The limit does not exist.

2. Find all the roots of the equation $\sin(z) = \cosh(4)$ by equating the real and imaginary parts of $\sin(z)$ and $\cosh(4)$.

   Solution:

$$\sin(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\sin(z) = \cosh(4)$$

   if and only if

$$\sin(x)\cosh(y) = \cosh(4)$$

   and

$$\cos(x)\sinh(y) = 0$$

   This occurs iff $\cos(x) = 0$ or $\sinh(y) = 0$. This occurs iff

$$x = n + \pi/2$$

   (in other words $\sin(x) = (-1)^n$) or $e^{2y} = 1$. If $\cos(x) = 0$, we can solve $\sin(x)\cosh(y) = \cosh(4)$ by $\cosh(y) = (-1)^n \cosh(4)$ but $\cosh(y) \geq 0$ so $\cosh(y) = \cosh(4)$ and n is even.

$$e^y + e^{-y} = e^4 + e^{-4}$$

   if and only if

$$y = \pm 4$$

   Also $x = \pi/2 + 2\pi n, n \in \mathbf{Z}$

3. Express each of the following in polar coordinates: i, $1-i$, $\sqrt{3}-i$.

Solution: $i = e^{i\pi/2}$

$1-i = \sqrt{2}e^{-i\pi/4}$

$\sqrt{3}-i = 2e^{-i\pi/6}$

4. Express each of the following as $x+iy$: $e^{4\pi i/3}, e^{5\pi i/6}, (1+i)^{-3}$

Solution: (a) $e^{4\pi i/3} = e^{i\pi/3} = -(1/2+\sqrt{3}/2i)$

(b) $e^{5\pi i/6} = e^{6i\pi/6}e^{-i\pi/6} = -e^{-i\pi/6} = -(\sqrt{3}/2 - i/2) = -\sqrt{3}/2 + i/2$

(c)

$$(1+i)^{-3} = \left(\frac{1}{1+i}\right)^3$$

$$= \left(\frac{1-i}{(1+i)(1-i)}\right)^3 = \frac{1}{8}(1-i)^3$$

$$= \frac{1}{8}(1-3i+3i^3-i^3) = \frac{1}{8}(-2-2i) = \frac{1}{4}(-1-i)$$

5. Describe each of the following sets geometrically. Which are open, which are closed, and which are compact?

(i) $\{z : |z-1-i| = 1\}$

(ii) $\{z : |z-1+i| \geq |z-1-i|\}$

(iii) $\{z : |z+i| \neq |z-i|\}$

(iv) $\{z = |z|e^{i\theta} : \pi/4 < \theta < 3\pi/4\}$

Solutions:

(i) A circle, centre $1+i$, radius 1: Closed, compact, not open.

(ii)

$$\{z \mid |z-1+i| \geq |z-1-i|\}$$

The set of points closer to $1+i$ than to $1-i$, or of equal distance to both, is the closed upper half plane $\{z|\mathrm{Im}(z) \geq 0\}$. This is closed, not open, not compact.

(iii)

But $\{z : |z+i| = |z-i|\}$ is the line of points of equal distance from the points i and $-i$, which is the real axis. In other words it is $\mathbf{C} \setminus \{z : \mathrm{Im}(z) = 0\}$ This is open, not closed, not compact.

(iv) $\{z = |z|e^{i\theta} : \pi/4 < \theta < 3\pi/4\}$ This is the set of points with polar angle $> \pi/4$ and $< 3\pi/4$. This is open, not closed, not compact.

6. For each of the following choices of f, either obtain $\lim_{z\to 0} f(z)$ or prove that the limit does not exist. (i) $|z|^2/z$,

   (ii) $\bar{z}/z$

   Solution: (i) $\lim_{z\to 0} |z|^2/z = \lim_{z\to 0} \bar{z} = 0$ since $|z|^2 = z\bar{z}$.

   (ii)

$$\lim_{z\to 0} \bar{z}/z = \lim_{z\to 0} \frac{re^{-i\theta}}{re^{i\theta}} = \lim_{z\to 0} e^{-2i\theta}$$

   This limit does not exist.

7. Prove that f is continuous on $\mathbf{C}$ when (i) $f(z) = \bar{z}$

   (ii) $f(z) = \mathrm{Im}(z)$

   (iii) $f(z) = \mathrm{Re}(z^3)$

   Solution:

   $f(z) = \bar{z}$

   $f(z+h) = \bar{z} + \bar{h} \to \bar{z}$ as $h \to 0$.

   So f is continuous on $\mathbf{C}$.

   (ii) $f(z) = \mathrm{Im}(z)$

   $f(z+h) = \mathrm{Im}(z) + \mathrm{Im}(h) \to \mathrm{Im}(z)$ as $h \to 0$ so f continuous on $\mathbf{C}$

   (iii)

$$f(z) = \mathrm{Re}(z^2) = \frac{z^2 + \bar{z}^2}{2}$$

$$f(z+h) = \mathrm{Re}(z+h)^2 = \frac{1}{2}\left((z+h)^2 + (\bar{z}+\bar{h})^2\right)$$

$$= \frac{1}{2}\left(z^2 + \bar{z}^2 + 2hz + 2\bar{h}\bar{z} + h^2 + \bar{h}^2\right) \to \frac{1}{2}\left(z^2 + \bar{z}^2\right)$$

   as $h \to 0$, so f is continuous on $\mathbf{C}$.

8. Prove that f defined by $f(z) = z^5/|z|^4$ $(z \neq 0)$, $f(0) = 0$ satisfies the Cauchy-Riemann equations at $z = 0$ but is not differentiable there.

Proof: $f(z) = z^5/|z|^4$ $(z \neq 0)$, $f(0) = 0$

$$= z^5 / \left(z^2 \bar{z}^2\right)$$

$$= z^3 \bar{z}^{-2}$$

$$= \frac{(x+iy)^5}{(x^2+y^2)^2} = \frac{1}{(x^2+y^2)^2}\left(x^5 - 10x^3y^2 + +5xy^4 + i(5x^4y - 10x^2y^3 + y^5)\right)$$

$$= u(x,y) + iv(x,y)$$

where

$$u = \frac{1}{(x^2+y^2)^2}\left(x^5 - 10x^3y^2 + 5xy^4\right)$$

$$v = \frac{1}{(x^2+y^2)^2}\left(5x^4y - 10x^2y^3 + y^5\right)$$

$$u_x = \left(5x^4 - 30x^2y^2 + 5y^4\right)\frac{1}{(x^2+y^2)^2}(:= A)$$

$$-\frac{4x}{(x^2+y^2)^3}\left(x^5 - 10x^3y^2 + 5xy^4\right)$$

$$v_y = \left(5x^4 - 30x^2y^2 + 5y^4\right)\frac{1}{(x^2+y^2)^2} - \frac{4y}{(x^2+y^2)^3}\left(5x^4y - 10x^2y^3 + y^5\right)$$

$$v_x = \left(20x^3y - 20xy^3\right)\frac{1}{(x^2+y^2)^2}(:= B)$$

$$-\frac{4x}{(x^2+y^2)^3}\left(5x^4y - 10x^2y^3 + y^5\right)$$

$$-u_y = -\left(20x^3y - 20xy^3\right)\frac{1}{(x^2+y^2)^2} + \frac{4y}{(x^2+y^2)^3}\left(x^5 - 10x^3y^2 + 5xy^4\right)$$

To evaluate u_x and v_y when $x = 0$ and $y = 0$,

$$u_x(0,0) = \lim_{x \to 0}\frac{u(x,0) - u(0,0)}{x} = \left(\frac{d}{dx}x^5/x^4\right)_{x=0} = \left(\frac{d}{dx}x\right)_{x=0} = 1.$$

$$v_y(0,0) = \frac{d}{dy}\,|_0\; v(0,y) = \left(\frac{d}{dy}y^5/y^4\right)_{y=0} = \left(\frac{d}{dy}y\right)_{y=0} = 1$$

Similarly

$$u_y(0,0) = \left(\frac{d}{dy}u(0,y)\right)_{y=0} = \left(\frac{d}{dy}(0)\right)_{y=0} = 0$$

$$-v_x(0,0) = \left(\frac{d}{dx}v(x,0)\right)_{x=0} = \left(\frac{d}{dx}(0)\right)_{x=0} = 0$$

Thus $u_x(0,0) = v_y(0,0)$ and $v_x(0,0) = -u_y(0,0)$ So f satisfies the Cauchy-Riemann equations when $x = 0$ and $y = 0$.

Claim f is not differentiable at 0.

$$A = \frac{1}{h}\left(\frac{(z+h)^5}{|z+h|^4} - z^5/|z|^4\right) = \frac{\bar{h}}{h}\left(\frac{3|z|^4}{\bar{z}^4} + \frac{\bar{h}}{h}\frac{-|z|^2z^2}{\bar{z}^4}\right)$$

plus higher order in h.

So $\lim_{h\to 0} A$ does not exist because $\lim_{h\to 0} \bar{h}/h$ does not exist. Hence f is not differentiable.

A cleaner solution may be obtained by setting $z = 0$ at the beginning:

$$A = \frac{1}{h}\left(\frac{h^5}{\bar{h}^4}\right) = \frac{h^3}{\bar{h}^2}.$$

This depends on θ (where $h = re^{i\theta}$) so the derivative does not exist.

9. Which of the following are holomorphic?

(i) $e^z/z(z-1)(z-2)$

(ii) $(1+e^z)^{-1}$

Solution:

(i) $f(z) = \frac{e^z}{z(z-1)(z-2)}$ $g(z) = e^z$ is holomorphic everywhere
$h(z) = \frac{1}{z(z-1)(z-2)}$ is holomorphic when $z \neq 0, 1, 2$
So $f = gh$ is holomorphic when $z \neq 0, 1, 2$

(ii) **Solution:** $f(z) = \frac{1}{(1+e^z)}$

$$1 + e^z \neq 0 \iff e^z \neq -1$$

$$\begin{aligned} &\iff e^z e^{i\pi} \neq 1 \\ &\iff z + i\pi \notin 2\pi i\mathbf{Z} \\ &\iff z \neq i\pi(2n+1) \end{aligned}$$

for any integer n

10. Where do the following series define holomorphic functions?

    (i) $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!}$ **Solution:**

    This is the Taylor series for $\exp(-z) - 1$. It has radius of convergence ∞.

    (ii)

    $$\sum_{n=0}^{\infty} z^{5n}$$

    Solution:

    $$\sum_{n=0}^{\infty} z^{5n} = \sum_{n=0}^{\infty} (z^5)^n = \frac{1}{1 - z^5}$$

    By the Ratio Test, the series converges when $|z| < 1$, diverges when $|z| > 1$ (since the ratio of terms is $|a_n/a_{n-1}| = |z^5|$). So the series defines a holomorphic function iff $|z| < 1$.

11. Determine for which values of z the following series converge absolutely:

    (i) $\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^n}$

    (ii) $\sum_{n=0}^{\infty} \left(\frac{z-1}{z+1}\right)^n$

    Solution:

    (i)

    $\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{z+1}{2}\right)^n$. By the Ratio Test this series converges absolutely when $|\frac{z+1}{2}| < 1$ and diverges when $\frac{z+1}{2} > 1$. This means it converges absolutely when $|z+1| < 2$.

    (ii)

    $\sum_{n=0}^{\infty} \left(\frac{z-1}{z+1}\right)^n$ By ratio test, series converges absolutely iff $\frac{z-1}{z+1} < 1$ iff $|z-1| < |z+1|$ iff $\mathrm{Re}(z) > 0$.

12. Write down an expansion of the form $\sum_{n=0}^{\infty} c_n z^n$ for

(i) $\frac{1}{1+z^4}$

(ii) $\frac{1}{1+z+z^2}$

Solution:

(i) $\frac{1}{1+z^4} = \sum_{n=0}^{\infty}(-1)^n z^{4n}$ is valid when $|z| < 1$.

(ii) $\frac{1}{1+z+z^2} = \sum_{n=0}^{\infty}(-1)^n A^n$ where $A = z(1+z)$ (by binomial expansion)

$$= \sum_{n=0}^{\infty}(-1)^n z^n (1+z)^n$$

Define $w = e^{2\pi i/3}$ so $\bar{w} = w^{-1}$. Then

$$\frac{1}{z^2+z+1} = \frac{1}{(z-w)(z-\bar{w})}$$

where $z^2+z+1 = 0$ iff $z = \frac{1}{2}(-1 \pm \sqrt{3}i) = e^{\pm 2\pi i/3}$

Then $\frac{1}{(z-w)(z-\bar{w})} = \frac{1}{(1-z/w))(1-z/\bar{w})} = \sum_{n\geq 0}\sum_{m\geq 0} z^n w^{-n} \sum_{m\geq 0} z^m w^m$
$= \sum_{n\geq 0, m\geq 0} z^{n+m} w^{m-n}$.

Let $n+m = k$ and $|m-n| \leq k$, $m-n = k-2n$.

So we have

$$\sum_{k\geq 0} z^k \sum_{l=k \mod 2, 0\leq |l|\leq k} w^l$$

So $w^{-k} + w^{-k+2} + \ldots + w^{k-2} + w^k = w^{-k}(1 + w^2 + \ldots + w^{2k})$
$= w^{-k}\left(\frac{1-w^{2k+2}}{1-w^2}\right) = \frac{w^{-k}-w^{k+2}}{1-w^2}$.

So the power series is $\sum_{k\geq 0} z^k w^{-k}\left(\frac{1-w^{2k+2}}{1-w^2}\right)$

13. Find all solutions of $\cos^2 z = 4$.

Solution:

$\cos^2 z = 4$ iff $\cos z = \pm 2$ iff $e^{iz} + e^{-iz} = \pm 4$ iff $A = e^{iz}$, $A^2 \mp 4A + 1 = 0$
$A = \frac{1}{2}\left(4 \pm \sqrt{12}\right)$.

In other words $A = 2 \pm \sqrt{3}$ or $A = -2 \pm \sqrt{3}$.

This is true iff $e^{iz} = 2 \pm \sqrt{3}$ or $e^{iz} = -2 \pm \sqrt{3}$

so

$iz = \ln(2 \pm \sqrt{3})$ or $iz = \ln(-2 \pm \sqrt{3})$

So

$$z = -i\left(\log_{\mathbf{R}}(2+\sqrt{3}) + 2\pi i n\right)$$

or

$$z = -i\left(\log_{\mathbf{R}}(-2+\sqrt{3}) + 2\pi i n\right)$$

or

$$z = -i\left(\log_{\mathbf{R}}(2+\sqrt{3}) + i\pi + 2\pi i n\right)$$

or

$$z = -i\left(\log_{\mathbf{R}}(-2+\sqrt{3}) + i\pi + 2\pi i n\right)$$

for $n \in \mathbf{Z}$.