University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/19

Final

Thursday, December 20, 2018, 7:00 pm –10:00 pm

FAMILY NAME: ______________________________

GIVEN NAMES: ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE: ______________________________

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

FOR MARKERS ONLY	
Question	Marks
1	/ 10
2	/ 15
3	/ 15
4	/ 15
5	/ 15
6	/ 15
7	/ 15
TOTAL	**/100**

No books or calculators may be used

You may use any theorems stated in class, as long as you state them clearly and correctly.

(1) (10 pts) (a) State the Cauchy-Riemann equations.

Solution:

$$u_x = v_x, v_x = -u_y$$

(b) Let $f(x, y)$ be a complex-valued function on the complex plane. Show that if $\partial f/\partial y = 0$ for all x and y then f is constant.

Solution:

$f_y = 0$ so f=u(x,y) + iv(x,y) then $u_y = v_y = 0$. By C-R, also $v_x = u_x = 0$ This implise $f = const$ because $\partial f/\partial x = \partial f/\partial y = 0$.

(2) (15 pts) Compute

$$\int_{|z|=2} \frac{dz}{(z-1)(z-i)^2}.$$

The line integral is around a circle of radius 2 and center 0 in the complex plane.

Solution:

This function has two poles, one at $z = 1$ and the other at $z = i$. The residue at $z = 1$ is $1/(1-i)^2$. The residue at $z = i$ is $h'(i)$ where $h(z) = (z-1)^{-1}$. So thhe residue at $z = i$ is $-1/(i-1)^2$. Hence the integral is

$$2\pi i(1/(1-i)^2 - 1/(1-i)^2) = 0.$$

(3) (15 pts)

(a) (8 points) Find the Laurent series of $\frac{1}{(z+1)^2}$ around 0. What it its radius of convergence ? Solution:

$$1/(z+1)^2 = -d/dz(1/(1+z))$$
$$1/(1+z) = 1 - z + z^2 - z^3 + \dots$$
$$1/(1+z)^2 = 1 - 2z + 3z^2 - \dots$$

Radius of convergence 1.

(b) (7 pts) Find the Laurent series of $\frac{1}{z+1}$ around 11. What is its radius of convergence?

Solution:

$$1/(z+1) = 1/(z-1+2) = (1/2))1/(1+2/(z-1)) = (1/2)(1-2/(z-1)+(2/(z-1)^2)-\dots$$

The radius of convergence is 1.

(4) (15 pts)

Compute the integral

$$\int_\gamma z^n(1-z)^m dz$$

where m is a nonnegative integer and n is an integer. The curve γ is a circle of radius 2 and center 0 in the complex plane.

Solution:

$$(1-z)^m = \sum_{k=0}^{m}(mchoosek)(-z)^k$$

$$\int_\gamma z^n(1-z)^m dz = \sum_{k=0}^{m}(mchoosek)(-1)^k z^{k+n} dz$$

This is only nonzero if $k+n=-1$, in which case the integral is $2\pi i$. So the answer is

$$2\pi i(mchoosen-1).$$

(5) (15 pts) Use the Cauchy integral formula to compute the integral

$$\int_\gamma \frac{z^3+5}{z-i}dz$$

Here γ is a circle of radius 2 and center 0 in the complex plane.
Solution:

$$f(a) = \frac{1}{2\pi i}\int f(w)/(w-a)dw$$

so here

$$f(z) = z^3 + 5$$

and the integral is

$$2\pi i f(i)$$
$$= 2\pi i(-i+5) = 2\pi + 10\pi i$$

(6) (15 pts)

(a) Find the singularities of $\frac{\cos(z)}{\sin(z)}$. State the type of singularity (removable singularity, pole, essential singularity). If a pole, compute the order of the pole.

Solution: This function is singular when $\sin(z) = 0$, in other words when $z = n\pi$. Because $cos(z)$ is nonozero at those values and $\sin(z)$ has a zero of order 1 ($\sin(z) = (z - n\pi) + ...$), or the first derivative of $\sin(z)$ at these zeroes is nonzero), we find that

(b) Compute the residue of $\frac{\cos(z)}{\sin(z)}$ at $z = 0$.

Solution: Because the leading order term of $\sin(z)$ at $z = 0$ is z, and $\cos(0) = 1$, we find that the residue of this function at $z = 0$ is 1.

(7) (15 points)

Use residues to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}.$$

Solution: Complete the contour to a semicircle with radius R. Then

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} + \int_{\Gamma_R} f(z)dz = 2\pi i)\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=3i} f(z).$$

The residues are

$$Res_{z=i} f(z) = 1/(2i))(8)$$
$$Res_{z=3i} f(z) = 1/6i(-7)$$

The contour integral is

$$z = Re^{i\theta}$$
$$\frac{Re^{i\theta} i d\theta}{(R^2 e^{2i\theta}+1)(R^2 e^{2i\theta}+9)}$$

The absolute value of this is less than

$$\int \frac{Rd\theta}{(R^2-1)(R^2-9)}$$

which tends to 0 as $R \to \infty$.

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