Department of Computer & Mathematical Sciences
University of Toronto Scarborough

Final Examination
MATC34H – Complex Variables

Examiner: L. Jeffrey

Date: December 19, 2013
Time: 19:00 - 22:00

FAMILY NAME: ______________________________

GIVEN NAMES: ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE: ______________________________

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

- There are 8 numbered pages in this exam. It is your responsibility to ensure that, at the start of the exam, this booklet has all its pages.
- Answer all questions. Explain and justify your answers.
- No books or calculators are allowed.
- You may use any theorems stated in class, as long as you state them clearly and correctly.

FOR MARKERS ONLY

Question	Marks
1	/15
2	/12
3	/13
4	/15

1. **[15 marks]** At which points are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   At which points do the functions take the value 0?

   Identify the type of singularity (pole, removable singularity, essential singularity). Also identify the order of each pole and the order of each zero.

   (a) $\dfrac{z}{\sin(z)}$

   Answer: 0 is a removable singularity. $n\pi$ is a pole of order 1 for $n \neq 0$ an integer.

   (b) $\dfrac{\sin(z)}{z}$

   This function has no zeroes.

   Answer: 0 is a removable singularity.

   $n\pi, n \neq 0$ is a zero

   (c) $\dfrac{1}{e^z - 1}$

2. **[12 marks]** Compute

$$\int_\gamma \frac{1}{\sin(z)} dz$$

where

(a) γ is a circle of radius 4 and centre 0 oriented counterclockwise.

Solution: By the Cauchy residue theorem, this integral is $\sum_p \mathrm{Res}(f(x); p)$ where the sum is over points p inside the circle where the residue is nonzero.

In this case this means $p = 0, \pi, -\pi$.

So we need to compute the residue of $1/\sin(z)$ at these points.

At $z = 0$, $\sin(z) = z - z^3/3! + \ldots$ which leads to the residue of $1/\sin(z)$ at 0 being 1.

At $z = \pi$, we decompose $\sin(z) = -\sin(z - \pi) = -(z - \pi) + (z - \pi)^3/3! + ...$ which leads to the residue of $1/\sin(z)$ at $z = \pi$ being -1. Similarly the residue of this function at $z = -\pi$ is -1. This means the integral is $-2\pi i$.

(b) γ is a circle of radius 4 and centre 5 oriented counterclockwise.

Answer: In this case all the points πi where this function has nonzero radius are at a distance more than 4 away from the center of the circle, so the integral is 0 by Cauchy's residue theorem (or Cauchy's theorem).

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3. **[13 marks]** Compute the Laurent series of

$$\frac{1}{(z+1)}$$

at $z = 1$ which converges for

(a) $|z-1| < 2$.
Solution:

$$\begin{aligned}\frac{1}{z+1} &= \frac{1}{(z-1)+2} \\ &= \frac{1}{2}\frac{1}{1+w}\end{aligned}$$

where $w = (z-1)/2$. We expand $\frac{1}{1+w} = 1 - w + w^2 - \ldots$ using the binomial theorem. This converges for $|w| < 1$, or $|z-1| < 2$.

(b) $|z-1| > 2$
Solution:

$$\frac{1}{z+1} = \frac{1}{(z-1)+2} = \frac{1}{z-1}\frac{1}{1+v}$$

where $v = 2/(z-1)$. We now expand this using the binomial theorem (as a power series in v. This converges for $|v| < 1$, in other words for $|z-1| > 2$.

4. **[15 marks]** Is it possible for a holomorphic function f to take the values 1 when $z = \frac{1}{n}$ for n an even positive integer, and $f(z) = 0$ when $z = \frac{1}{n}$ for n an odd positive integer?

   If you claim yes, give an example; if you claim no, give a proof.

   Solution: By the identity theorem, any such function would have to equal 1 everywhere (because the points $1/2k$ for k a positive integer have a limit point). But by the same argument, this function must equal 0 everywhere (because the points $1/(2k+1)$ for positive integers k have a limit point.). Contradiction. So there can not exist any such function.

5. [**15 marks**] Use residue calculus to compute the following integral:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

You must explain how this integral is related to a contour integral. Compute any limits involved.

Solution: Let Γ_R be the semicircle $\{Re^{i\theta}, 0 \leq \theta \leq \pi\}$ in the upper half plane. Step 1: show that

$$|\int_{\Gamma_R} f(z)dz| \to 0$$

as $R \to \infty$. This integral is

$$|\int_0^{\pi} |\frac{Rie^{iR\theta}d\theta}{(R^2e^{2i\theta}+1)R^2e^{2i\theta}+4}|$$

$$\leq R\int_0^{\pi} d\theta \frac{1}{(R^2-1)(R^2-4)} \mapsto 0$$

as $R \to \infty$.

Step 2: Compute $\int_{\Gamma_R} f(z)dz + \int_{-R}^{R} f(z)dz$ where $f(z) = \frac{1}{(z^2+1)(z^2+4)}$.

Solution: By Cauchy's residue theorem , this integral is the sum of residues of $f(z)$ inside the closed contour that is the union of Γ_R and the interval $[-R, R]$. The only points where the function has nonzero residue are i and $2i$. The residue of $1/(z^2+1)(z^2+4)$ at i is $\frac{/1}{2i)(3)}$. Likewise the residue of this function at $2i$ is $\frac{1}{4i(-3)}$. The result is $\frac{1}{4i(3)} = -i/12$.

6. **[15 marks]**

   (a) Find a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

   for which

$$f(0) = 2$$
$$f(2) = i$$
$$f(1) = \infty$$

   Find the image of the real axis under this transformation.

   (b) Let $u(x, y) = x^2 - y^2$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.

N/A

7. **[15 marks]**

   (a) (5 points) State Cauchy's residue theorem.
   Solution: See question 2.

   (b) (10 points) Use Cauchy's residue theorem to compute the integral $\int_\gamma f(z)dz$ around a semicircle of radius 1 with centre 1 in the upper half plane, oriented counterclockwise, where $f(z)$ is the following function:

$$f(z) = \frac{1}{z^2 + 1/4}.$$

   Solution: This function has poles only at $z = \pm i/2$ The point $i/2$ satisfies $|i/2 - 1| > 1$, so it is outside the semicircle. So by Cauchy's residue theorem (or Cauchy's theorem), the integral is 0.

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