University of Toronto at Scarborough
Department of Computer and Mathematical Sciences, Mathematics

MAT C34F 2013/14

Final Examination

Thursday, December 19, 2013 ; 19:00–22:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(15 points)** At which points are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   At which points do the functions take the value 0?

   Identify the type of singularity (pole, removable singularity, essential singularity). Also identify the order of each pole and the order of each zero.

   (i)
   $$\frac{z}{\sin(z)}$$

   (ii)
   $$\frac{\sin(z)}{z}$$

   (iii)
   $$\frac{1}{e^z - 1}$$

2. **(12 points)**

   Compute
   $$\int_\gamma \frac{1}{\sin(z)} dz$$
   where

   (a) γ is a circle of radius 4 and centre 0 oriented counterclockwise.

   (b) γ is a circle of radius 4 and centre 5 oriented counterclockwise.

3. **(13 points)**

   Compute the Laurent series of

   $$\frac{1}{(z+1)}$$

   at $z = 1$ which converges for

   (a) $|z-1| < 2$.

   (b) $|z-1| > 2$

4. **15 points** Is it possible for a holomorphic function f to take the values 1 when $z = \frac{1}{n}$ for n an even positive integer, and $f(z) = 0$ when $z = \frac{1}{n}$ for n an odd positive integer?

   If you claim yes, give an example; if you claim no, give a proof.

5. **(15 points)** Use residue calculus to compute the following integral:

   $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

   You must explain how this integral is related to a contour integral. Compute any limits involved.

6. **(15points)**

   (a) Find a Möbius transformation

   $$f(z) = \frac{az+b}{cz+d}$$

   for which

   $$f(0) = 2$$
   $$f(2) = i$$
   $$f(1) = \infty$$

   Find the image of the real axis under this transformation.

   (b) Let $u(x, y) = x^2 - y^2$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.

7. **(15 points)**

   (a) (5 points) State Cauchy's residue theorem.

   (b) (10 points) Use Cauchy's residue theorem to compute the integral $\int_\gamma f(z)dz$ around a semicircle of radius 1 with centre 1 in the upper half plane, oriented counter-clockwise, where $f(z)$ is the following function:

   $$f(z) = \frac{1}{z^2 + 1/4}.$$