University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2002/2003

Final Examination

Wednesday, December 11, 2002 ; 9:00–12:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(20 points)** At which values are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   Identify the type of singularity (pole, removable singularity, essential singularity).

   (i)
   $$\frac{e^z - 1}{z}$$
   Soln: $z = 0$ removable singularity

   (ii)
   $$\frac{1}{(z^2 - \pi^2)\sin^2(z)}$$
   Soln: $z = \pm\pi$ pole of order 1

   $z = n\pi$, n an integer $n \neq \pm 1$ pole of order 1

   (iii)
   $$\frac{1}{\sin 1/z}$$
   Soln: $z = 0$ essential singularity

2. **(15 points)**

   Compute
   $$\int_\gamma \frac{e^z}{(z^2+1)}$$
   where

(a) (a) γ is a circle of radius 2 and centre 0 oriented counterclockwise.

Soln: There are two poles $z = \pm i$. Both are inside this circle. So the integral is $2\pi i\,(\text{Res}_{z=i} f + \text{Res}_{z=-i} f)$ where $f(z) = \frac{e^z}{(z^2+1)}$. The residue of f at i is $e^i/(2i)$ while the residue of f at $-i$ is

$$-e^{-i}/(2i).$$

The sum of the two residues is $\sin(1)$ and the integral is $2\pi i \sin(1)$.

(b) (b) γ is a semicircle in the upper half plane with radius 2 and centre 0 oriented counterclockwise.

Soln: Only the pole at $z = i$ contributes. So the residue is $e^i/(2i)$ and the integral is πe^i.

3. **(15 points)**

(a) Compute the Laurent series of

$$\frac{1}{(z+1)}$$

which converges for

i. $|z-1| < 2$.
ii. $|z-1| > 2$

Soln:

$$z + 1 = (z-1) + 2$$

So

$$\frac{1}{z+1} = (1/2)\frac{1}{1+(z-1)/2}$$
$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n 2^{-n}$$

This converges for $|z-1| < 2$.

$$\frac{1}{z+1} = \frac{1}{[2(z-1)/2} \frac{1}{\frac{2}{z-1}+1}$$
$$= 1/(z-1) \sum_{n=0}^{\infty} (-1)^n (\frac{2}{z-1})^n$$

This converges for $|z-1| > 2$.

(b) Compute the Laurent series of

$$\frac{1}{(z+1)^2}$$

which converges for $|z-1| < 2$.

Soln: $\frac{1}{(z+1)^2} = -(d/dz)(1/(z+1))$ so we differentiate the series

$$-(d/dz)\sum_{n=0}^{\infty}(-1)^n(z-1)^n 2^{-n}$$

$$= -\sum_{n=0}^{\infty}(-1)^n \frac{n}{2^n}(z-1)^{n-1}$$

4. **(15 points)** Use residue calculus to compute the following integral:

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2+2x+2)^2}$$

Soln: By methods discussed in class, we complete to a contour integral around a semicircle centre 0 radius R in the upper half plane. As we showed in class, the limit of the integral over the semicircle is 0 as $R \to \infty$ (because the degree of the polynomial in the denominator is 4 and the degree of the polynomial in the numerator is 1, so the difference is 3 which is greater than or equal to 2). So the integral is given by the sum of the residues in the upper half plane. These are at $(z+1)^2+1=0$ or $z+1=\pm i$ or $z=-1\pm i$ – the only one in the upper half plane is $z=-1+i$. To compute the residue, note that $z=-1+i+(z-(-1+i))$ so

$$\frac{z}{(z-(-1+i))^2(z-(-1-i))^2} = \frac{h(z)}{(z-a)^2}$$

where $a=-1+i$ and

$$h(z) = \frac{z}{(z-b)^2}$$

for $b=-1-i$. The residue is

$$h'(a) = \frac{1}{(a-b)^2} + (-2)\frac{a}{(a-b)^3}.$$

5. **(20 points)**

(a) Find a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

for which

$$f(0) = -1$$
$$f(1) = i$$
$$f(\infty) = 1$$

Find the images of the real axis and the imaginary axis under this transformation.
Soln:

$$b = -d$$
$$a + b = i(c + d)$$
$$a = c$$

so

$$(1 - i)a = (-1 - i)b$$
$$f(z) = \frac{z + (1 - i)/(-1 - i)}{z - (1 - i)/(-1 - i)}$$

$0, 1, \infty$ are in the real axis. So the circle containing $-1, i, 1$ is the image of the real axis. This is the circle with centre 0 and radius 1.

$0, i, \infty$ are in the imaginary axis. The image of i under f is ∞. So the image of the imaginary axis is the line containing $-1, 1, \infty$, which is the real axis.

(b) Let $u(x, y) = x^2 - y^2 + x + xy$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.

Soln:

$u_x = 2x + 1 + y = v_y$

so

$$v = h(x) + y^2/2 + (2x + 1)y$$
$$v_x = dh/dx + 2y = -u_y = -(-2y + x)$$

so

$$dh/dx = -x$$

and

$$h(x) = -x^2/2.$$

6. **(15 points)**

(a) State Cauchy's residue theorem.

(b) Use Cauchy's residue theorem to compute the integral $\int_\gamma f(z)dz$ around a circle of radius $1/4$ with centre i, oriented counterclockwise, where $f(z)$ is the following function:

$$f(z) = \frac{1}{z^2 - z^6}.$$

Soln:

$$f(z) = -\frac{1}{z^2(z - 1)(z + 1)(z - i)(z + i)}$$

So the residue at $z = i$ is

$$\frac{1}{(i-1)(i+1)(2i)}$$

This is the only pole inside the circle.