University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2002/2003

Final Examination

Wednesday, December 11, 2002 ; 9:00–12:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(20 points)** At which values are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   Identify the type of singularity (pole, removable singularity, essential singularity).

   (i)
   $$\frac{e^z - 1}{z}$$

   (ii)
   $$\frac{1}{(z^2 - \pi^2)\sin^2(z)}$$

   (iii)
   $$\frac{1}{\sin 1/z}$$

2. **(15 points)**

   Compute
   $$\int_\gamma \frac{e^z}{(z^2+1)}$$
   where

   (a) (a) γ is a circle of radius 2 and centre 0 oriented counterclockwise.

   (b) (b) γ is a semicircle in the upper half plane with radius 2 and centre 0 oriented counterclockwise.

3. **(15 points)**

(a) Compute the Laurent series of

$$\frac{1}{(z+1)}$$

which converges for

i. $|z-1| < 2$.

ii. $|z-1| > 2$

(b) Compute the Laurent series of

$$\frac{1}{(z+1)^2}$$

which converges for $|z-1| < 2$.

4. **(15 points)** Use residue calculus to compute the following integral:

$$\int_{-\infty}^{\infty} \frac{x dx}{(x^2+2x+2)^2}$$

5. **(20 points)**

(a) Find a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

for which

$$f(0) = -1$$
$$f(1) = i$$
$$f(\infty) = 1$$

Find the images of the real axis and the imaginary axis under this transformation.

(b) Let $u(x, y) = x^2 - y^2 + x + xy$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.

6. **(15 points)**

(a) State Cauchy's residue theorem.

(b) Use Cauchy's residue theorem to compute the integral $\int_\gamma f(z)dz$ around a circle of radius $1/4$ with centre i, oriented counterclockwise, where $f(z)$ is the following function:

$$f(z) = \frac{1}{z^2 - z^6}.$$