University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2001/2002

Final Examination

Friday, December 14, 2001 ; 2:00–5:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(20 points)** At which values are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   Identify the type of singularity (pole, removable singularity, essential singularity).

   (i)
   $$\frac{1}{(z-1)\sin^2(z)}$$

   Solution: $z = 1$ pole of order 1

   $z = n\pi$, n an integer: pole of order 2

   (ii)
   $$\frac{e^{1/z}}{z^2}$$

   Soln: $z = 0$ essential singularity.

   (iii)
   $$\frac{z}{e^z - 1}$$

   Soln: $z = 0$ removable singularity

2. **(15 points)**

   Compute
   $$\int_\gamma \frac{e^{2z}}{(z-1)^3}$$
   where γ is a circle of radius 2 and centre 0 oriented counterclockwise.

   Soln:

$$e^{2z} = e^2 e^{2(z-1)} = e^2 \sum_{m=0}^{\infty} \frac{2^m (z-1)^m}{m!}$$

so the residue comes from the term with $m = 2$ and the integral is $\frac{(4\pi i)}{e}^2$.

This question can also be done using Cauchy's integral formula:

Cauchy's integral formula says that

$$f''(a) = \frac{2}{2\pi i} \int_\gamma \frac{e^{2z}}{(z-1)^3} dz$$

where

$$f(z) = e^{2z}$$

for $a = 1$.

$$f''(1) = 4e^2$$

so the result of Cauchy's integral formula agrees with the above answer. "

3. **(15 points)** (i) State Liouville's theorem.

   Soln: Any function which is bounded and holomorphic everywhere in the complex plane is constant.

   (ii) Suppose f is holomorphic for all z in the complex plane. Suppose also that the k-th derivative $f^{(k)}(z)$ is bounded (in other words there is some number C such that $|f^{(k)}(z)| \leq C$ for all z).

   Show that f is a polynomial of degree less than or equal to k.

   Soln:

$$f^{(k)}(z)$$

   is a bounded holomorphic function of z, everywhere in the complex plane. So

$$f^{(k)}(z)$$

   is constant. Integrating $f^{(k)}$ with respect to z, it follows that f is a polynomial of degree $\leq k$.

4. **(15 points)** (i) Compute the Laurent series of

$$\frac{z}{(z+2)^2}$$

   at $z = 0$.

Soln: $-2\frac{1}{(z+2)^2} = \frac{d}{dz}\frac{1}{z+2}$.

$$\frac{1}{z+2} = (1/2)\sum_{n=0}^{\infty}(-1)^n(z/2)^n$$

so

$$-2\frac{1}{(z+2)^2} = \frac{d}{dz}\frac{1}{z+2} = \sum_{n=0}^{\infty}(-1/2)^n n z^{n-1}$$

and

$$\frac{z}{(z+2)^2} = (-1/2)\sum_{n=0}^{\infty}(-1/2)^n n z^n.$$

(ii) Give the three terms corresponding to the lowest powers of z in the Laurent series at 0 of the following function:

$$f(z) = \frac{1}{e^z - 1}$$

Soln:

$$e^z - 1 = z(1 + z/2 + z^2/6 + \ldots)$$

so using

$$(1 + ax + bx^2)^{-1} = (1 - (ax + bx^2) + (ax + bx^2)^2/2 = 1 - (ax + bx^2) + (a^2/2)x^2$$
$$= 1 - ax + (a^2/2 - b)x^2$$

(to order x^2) we get (using $a = 1/2, b = 1/6$) that the first three terms in the Laurent series are

$$(e^z - 1)^{-1} = z^{-1}\left(1 - z/2 + (1/8 - 1/6)z^2\right)$$

5. **(15 points)** Use residue calculus to compute the following integral:

$$\int_{-\infty}^{\infty}\frac{dx}{x^4 + x^2 + 1}$$

Soln: See 2000 exam, question 6

6. **(20 points)**

(i) Find a Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

for which

$$f(0) = 2$$

$$f(1) = i$$

$$f(\infty) = 0$$

Soln:

$$b = 2d$$

$$a + b = i(c + d)$$

$$a = 0$$

$$b = 2d = i(c + d)$$

so

$$ic = (2 - i)d$$

so

$$-c = (2i + 1)d$$

So

$$f(z) = \frac{2d}{(-2i - 1)dz + d} = \frac{2}{(-2i - 1)z + 1}.$$

(ii) Let $u(x, y) = e^x \cos(y)$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.

Soln:

$$u_x = u = e^x \cos(y)$$

We set

$$u_x = v_y$$

so

$$v_y = e^x \cos(y)$$

so

$$v = e^x \sin(y) + h(x)$$

so

$$v_x = e^x \sin(y) + \frac{dh}{dx} = -u_y = e^x \sin(y)$$

so

$$dh/dy = 0$$

and

$$h(x, y) = \text{const.}$$

So

$$v = e^x \sin(y)$$