University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2000/2001

Final Examination

Friday, December 15, 2000 ; 9:00 - 12:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(15 points)** Compute

$$\int_\Gamma \frac{\sin(z)dz}{(2z-\pi)^2(z-2\pi)}$$

where Γ is the circle with centre 0 and radius π.

2. **(15 points)** Find the singularities of the following functions. For each singularity, state whether it is a removable singularity, a pole or an essential singularity, and justify your statement. Identify the orders of all poles.

You need not consider whether the point at infinity is a singularity.

(a)
$$\frac{1}{(z^2+4)\sin(\pi z)}$$

(b)
$$\frac{z}{\sin(\pi z)}$$

(c)
$$\sin(1/z)$$

3. **(15 points)**

(a) Using residues, find the integral

$$\int_\gamma \frac{e^{z^3}dz}{(z-1)^2}$$

where γ is the circle with centre 0 and radius 3.

(b) Let Γ be the square with vertices $+1+i$, $-1+i$, $-1-i$ and $+1-i$ traversed in that order (in other words counterclockwise). Compute the integral

$$\int_\Gamma \frac{\cos(z)dz}{z^3}.$$

4. **(15 points)**

(a) Find the Laurent series at 0 for

$$\frac{\sin(z)}{z^2}.$$

(b) Find the first three nonzero terms in the Laurent series at 0 for

$$\frac{z^2}{\sin(z)}.$$

5. **(15 points)** Find a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

for which

$$f(0) = 0$$
$$f(1) = \infty$$
$$f(\infty) = 1$$

Find the image under f of the real axis and the imaginary axis. If the image is a line, you should give the line in terms of the direction perpendicular to it and a point on it. If the image is a circle, you should state the centre and the radius of this circle.

6. **(15 points)** Use contour integrals to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1}.$$

7. **(10 points)** Which of the following statements are true, and which are false? If a statement is true because of a theorem stated or proved in class, you should give the name of the theorem or state it (you are not required to prove it). If a statement is not true, you should give an example which shows it isn't valid.

(1.) If $\int_\triangle f dz = 0$ for all triangles $\triangle$ in a region G, then f is holomorphic in G.

(2.) If f is a holomorphic function on the complex plane, then f is bounded.

(3.) If f is a bounded holomorphic function on the complex plane, then f is constant.

(4.) If γ is a simple closed curve and $\int_\gamma f dz = 0$ then f is holomorphic inside γ.

(5.) If f is holomorphic inside a simple closed curve γ, then $\int_\gamma f dz = 0$

(6.) If f is holomorphic on the unit disc $\{z \mid |z| \leq 1\}$ then the minimum value of $|f|$ occurs on the boundary $\{z \mid \ |z| = 1\}$.

(7.) If f is holomorphic on the unit disc $\{z \mid |z| \leq 1\}$ then the maximum value of $|f|$ occurs on the boundary $\{z \mid \ |z| = 1\}$.

(8.) There is a complex number z for which $\cos^2(z) + \sin^2(z) = 2$

(9.) The integral $\int_\gamma \frac{1}{z} dz$ has the same value for any simple closed curve γ for which $\gamma(t)$ is not equal to 0 for any t.

(10.) It is possible to define a continuous holomorphic function f on a region containing the unit circle $\{z| \ |z| = 1\}$ such that $f(z)$ is equal to one of the values of $\sqrt{z}$ everywhere on the unit circle.

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