MATC34 2013 Solutions to Assignment 5

1.
$$\int_{\gamma(0;1)} \frac{z}{2z^4+5z^2+2}dz$$
$$= \int_0^{2\pi} \frac{e^{i\theta} i e^{i\theta} d\theta}{2e^{4i\theta}+5e^{2i\theta}+2}$$
$$= i\int_0^{2\pi} \frac{d\theta}{2e^{2i\theta+5+2e^{-2i\theta}} = i\int_0^{2\pi} \frac{d\theta}{8\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2-4+5}}$$
$$= i\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta+1}$$

To compute
$$\int_{\gamma(0;1)} \frac{zdz}{2z^4+5z^2+2}$$

This has poles when
$$z^4+5/2z^2+1=0$$

in other words when $z^2=-2$ or $z^2=-1/2$, or $z=\pm\sqrt{2}i$ or $z=\pm\sqrt{2}^{-1}i$

We are only interested in the poles at $\pm\sqrt{2}^{-1}i$ since only these are inside $\gamma(0;1)$

$$f(z) = \frac{z}{2z^4+5z^2+2} = \frac{1}{2(z-\sqrt{2}i)(z+\sqrt{2}i)(z-\sqrt{2}^{-1}i)(z+\sqrt{2}^{-1}i)}$$

So
$$\text{Res}_{z=\sqrt{2}^{-1}i} f(z) = \frac{\sqrt{2}^{-1}i}{2(-1/2+2)(\sqrt{2}i)}$$

$$\text{Res}_{z=-\sqrt{2}^{-1}i} f(z) = -\frac{\sqrt{2}^{-1}i}{2(-1/2+2)(-\sqrt{2}i)}$$

So
$$\sum_b \text{Res}_{z=b} f(z) = \frac{i}{\sqrt{2}\cdot 3\cdot\sqrt{2}i}2 = 1/3.$$

So $i\int_0^{2\pi}\frac{d\theta}{8\cos^2\theta+1} = \frac{2\pi i}{3}/$

This shows $\int_0^{2\pi}\frac{d\theta}{8\cos^2\theta+1} = 2\pi/3$.

2.
$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}dz}{(z^4+z^3+z^2+z+1)^2}$$
$$=\lim_{R\to\infty}\int_0^{\pi}\frac{e^{iR(\cos\theta+i\sin\theta)}iRe^{i\theta}d\theta}{\left(R^4e^{4i\theta}+R^3e^{3i\theta}+R^2e^{2i\theta}+Re^{i\theta}+1\right)^2}$$
$$=\lim_{R\to 0}\int_0^{\pi}\frac{e^{-R\sin\theta}e^{iR\cos\theta}Re^{i\theta}d\theta}{R^8\left(e^{4i\theta}+\frac{1}{R}e^{3i\theta}+\frac{1}{R^2}e^{2i\theta}+\frac{1}{R^3}e^{i\theta}+\frac{1}{R^4}\right)}$$
$$=\lim_{R\to\infty}\int_0^{\pi}\frac{f(\theta)d\theta}{g(\theta)}$$

so
$$\mid\lim_{R\to\infty}\int_0^{\pi}\frac{f_R(\theta)d\theta}{g_R(\theta)}\mid\leq\lim_{R\to\infty}\int_0^{\pi}\mid\frac{f(\theta)}{g(\theta)}\mid d\theta$$

and $f_R(\theta)=ie^{-R\sin\theta}e^{iR\theta}Re^{i\theta}$ so $|mid f_R(\theta)\mid\leq$R since $e^{-R\sin\theta}\leq 1$ while
$$g_R(\theta)=R^8\left(e^{4i\theta}+\frac{1}{R}e^{3i\theta}+\frac{1}{R^2}e^{2i\theta}+\frac{1}{R^3}e^{i\theta}+\frac{1}{R^4}\right)^2$$

hence $|g_R(\theta)|\geq R^8\left(1-1/R-1/R^2-1/R^3-1/R^4\right)\geq R^8/2$ for large R.

So $\mid\frac{f_R(\theta)}{g_R(\theta)}\leq 2/R^7$. So $\lim_{R\to\infty}\int_{\Gamma_R}\frac{f_R(\theta)}{g_R(\theta)}d\theta=0$.

3. Suppose f is holomorphic in $\mathbf{C}$ except for double poles at 1 (resp. -1) of residues a (resp. b) and $|z^2f(z)|\leq K$ for large $|z|$.

For any contour $\Gamma(0;R)$ where $R>|a|$, $R>|b|$ we have
$$\int_{\Gamma(0;R)}f(z)dz=2\pi i(a+b)$$

But
$$\mid\int_{\Gamma(0;R)}f(z)dz\mid\leq\int_{\Gamma(0;R)}|f(z)|dz$$

and $|f(z)|\leq K/R^2$ when $|z|=R$ so
$$\int_{\Gamma(0;R)}|f(z)|dz\leq K/R^2(2\pi R)$$

So $|\int f(z)dz| = 0$ since R is arbitrary. So $a+b=0$.

If $|z^2 f(z)| \leq K$,

$$f(z) - A/(z-1)^2 - B/(z+1)^2 - a/(z-1) - b/(z+1)$$

is holomorphic on $\mathbf{C}$ if the Laurent coefficients of f at ± 1 are A and B (in other words the Laurent series at 1 is $A/(z-1)^2 + a/(z-1)$ and the Laurent series at -1 is $B/(z+1)^2 + b/(z+1)$). Thus

$$z^2 f(z) - Az^2/(z-1)^2 - Bz^2/(z+1)^2 - az^2/(z-1)^2 - bz^2/(z+1)$$

is holomorphic everywhere Heence it is also bounded. So by Liouville it is a constant C.

$$z^2 f(z) = Az^2/(z-1)^2 + Bz^2/(z+1)^2 + az^2/(z-1)^2 + bz^2/(z+1) + C$$

$a = 1$: Substitute $z = 0$ and conclude $C = 0$. Since $a = 1$ and $f(2i) = f(-2i) = 0$

$$0 = A\frac{-4}{(2i-1)^2} + B\frac{(-4)}{(2i+1)^2} + \frac{-4}{2i-1} \qquad (1)$$

$$0 = A\frac{-4}{(-2i-1)^2} + B\frac{(-4)}{(-2i+1)^2} + \frac{-4}{-2i-1} + \frac{4}{-2i-1} \qquad (2)$$

From equation 1,

$$\begin{aligned} 0 &= \frac{-4A(-2i-1)^2}{(2i-1)^2(-2i-1)^2} + \frac{-4B(-2i+1)^2}{(2i+1)^2(-2i+1)^2} + 4\left(\frac{-1}{2i-1} + \frac{1}{2i+1}\right) \\ &= \frac{-4A(2i+1)^2}{25} + \frac{-4B(-2i+1)^2}{25} + 4\frac{-2i-1+2i-1}{4+1} \\ &= \frac{4}{25}\left(-A(-3+4i) - B(-3-4i_+5(-2)\right) \end{aligned}$$

From equation (2),

$$0 = \frac{4}{25}\left(-A(-3-4i) - B(-3+4i) + 5(-2)\right)$$

(obtained from equation (1) by taking complex conjugate of all complex numbers except A and B)

So from (**??**)

$$0 = A(3-4i) + B(3+4i) - 10$$

$$0 = A(3+4i) + B(3-4i) - 10$$

Define $\omega = 3+4i$, $\bar{\omega} = 3-4i$ $|\omega|^2 = 25$ so $0 = 25A + \omega^2 B - 10\omega$

and

$$0 = 25A + \bar{\omega}^2 B - 10\bar{\omega}$$

Subtracting, $B(\omega^2 - \bar{\omega}^2) = 10(\omega - \bar{\omega})$ so $B = \frac{10}{\omega+\bar{\omega})} = 5/3$ From **??**,

$A\omega + B\bar{\omega} = 10$ so

$$A = \frac{10}{\omega + \bar{\omega} = 5/3}$$

4. To compute this we consider a semicircular contour $\tilde{\Gamma_R}$ of radius R and $\int_{\Gamma_R} f(z)dz$ where $f(z) = \frac{1}{(z^2+z+1)^2}$. First we show that if γ_R is the semicircular contour with radius R, $\gamma_R(\theta) = Re^{i\theta}$ $(0 \le \theta \le \pi)$

   Hence $\int_{\Gamma_R} f(z)dz \to 0$ as $R \to \infty$.

$$\mid \int_{\Gamma_R} f(z)dz \mid = \mid \int_{\theta=0}^{\pi} \frac{iRe^{i\theta}d\theta}{\left(R^2e^{2i\theta} + Re^{i\theta} + 1\right)^2} \mid$$

$$\le \frac{R\pi}{R^2 - R - 1)^2}$$

   so $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0$ Next we compute $\int_{\tilde{\gamma_R}} f(z)dz$.

   $f(z) = \frac{1}{(z-w)^2(z-\bar{\omega})^2}$ where $\omega = e^{2\pi i/3}$. So only the pole at ω is contained in $\tilde{\gamma_R}$

   Writing $f(z) = \frac{1}{(z-w)^2} h(z)$,

$$\mathrm{Res}_{z=w} f(z) = h'(w)$$

   Here

$$h(z) = \frac{1}{(z-\bar{w})^2}$$

$$h'(z) = -\frac{2}{(z-\bar{w})^3}$$

$$h'(w) = -\left(\frac{2}{(w-\bar{w})}\right)^3 = -\frac{2}{(2i\sin 2\pi/3)^3}$$

so

$$\mathrm{Res}_{z=w} f(z) = -\frac{8i}{12\sqrt{3}} = \frac{1}{4i(\sqrt{3}/2)^3}$$

so

$$2\pi i \mathrm{Res}_{z=w} f(z) = \frac{4\pi}{3\sqrt{3}}$$

so $\int_{\tilde{\gamma}_R} f(z)dz = \frac{4\pi}{3\sqrt{3}}$ Because $\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = 0$, we find that $\int_{-\infty}^{\infty} f(x)dx = \frac{4\pi}{3\sqrt{3}}$.

5. Prove $\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \pi/2$.

   (cf. Example 8.5)

   Integrate $f(z) := \frac{e^{2iz}-1}{z^2}$ around a semicircular contour (radius R, centre 0 with a smaller semicircle (radius ϵ, centre 0) excised

   Because $\sin^2(x) = \frac{(e^{ix}-e^{-ix})^2}{4}$,

$$\frac{\sin^2(x)}{x^2} = -\frac{\mathrm{Re}(e^{2ix}-1)}{x^2}$$

   So we compute $\int_{\tilde{\gamma}} f(z)dz$ where $f(z) = \frac{(e^{2iz}-1)}{z^2}$ and $\tilde{\gamma} = \Gamma_R + [-R, -\epsilon] - \Gamma_\epsilon + [\epsilon, R]$. Then

$$\mid \int_{\Gamma_R} f(z)dz \mid \leq \int \frac{\mid e^{2i(R\cos t + iR\sin t)} - 1 \mid Rdt}{R^2}$$

   (where $\Gamma_R(t) = Re^{it}, 0 \leq t \leq \pi$)

$$= \int_0^\pi \frac{\mid e^{2iR\cos t} e^{-2R\sin t} - 1 \mid Rdt}{R^2}$$

   $\leq 2\pi/R$ so $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0$.

   We examine $\int_{\Gamma_\epsilon} f(z)dz$: this is $\int_0^\pi \frac{\left(e^{2i\epsilon\cos(t)}e^{-2\epsilon\sin(t)}-1\right)e^{i(\epsilon e^{2t}} i\epsilon dt}{\epsilon^2}$ Using f has a simple pole as 0. The residue is computed as follows.

   $f(z) = \frac{e^{2iz}-1}{z^2} = \frac{e^{2iz}}{z^2} +$ (holomorphic in z)

so $\text{Res}_{z=0}f(z) = 2i = b$. Hence by

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)dz = ib(\theta_2 - \theta_1)$$
$$= i(2\pi i)$$

Hence $\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} f(z)dz = -2\pi$.

By Cauchy's theorem, since the only pole of f is at $z = 0$ which is outside the contour,

$$\int_{\Gamma_R} f(z)dz + \int_R^{-\epsilon} f(z)dz - \int_{\Gamma_\epsilon} f(z)dz + + \int_\epsilon^R f(z)dz = 0$$

so $2\int_\epsilon^R f(z)dz = \int_{\Gamma_\epsilon} f(z)dz - \int_{\Gamma_R} f(z)dz$ since f is an even function along the real axis.

Taking the limit as $\epsilon \to 0$, $2\int_0^R f(z)dz = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} f(z)dz = -2\pi$.

So $\int_0^\infty \frac{e^{2ix}-1}{x^2}dx = -\pi$.

$$\frac{\sin^2(x)}{x^2} = -\frac{1}{2}\text{Re}\left(\frac{e^{2ix}-1}{x^2}\right)$$

so $\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \pi/2$.

6. $\int_0^\infty \frac{(\ln(x))^2}{1+x^2}dx = \pi^3/8$. Put $f(z) = \frac{(\ln(z))^2}{1+z^2}$. $f$has poles at $\pm i$. Introduce a semicircular contour (the same as in previous exercise QQQQQ) So

$$\int_\epsilon^R \frac{(\ln(x))^2}{1+x^2}dx + \int_{\Gamma_R} f(z)dz + \int_{-R}^{-\epsilon} \frac{(\ln(x))^2}{1+x^2}dx - \int_{\Gamma_\epsilon} f(z)dz = 2\pi i \text{Res}_i f(z).$$

Now for $x \in \mathbf{R}$, $x, 0$, $x = |x|e^{i\pi}$ so $\ln(x) = \ln(|x|) + i\pi$.

Let $y = -x, y > 0$ So

$$\int_{-R}^{-\epsilon} \frac{(\ln(x))^2}{1+x^2}dx = \int_R^\epsilon \frac{(\ln(y)+i\pi)^2 d(-y)}{1+y^2}$$
$$= \int_\epsilon^R \frac{(\ln(y)+i\pi)^2 dy}{1+y^2}$$

$$\mid \int_{\Gamma_R} f(z)dz \mid \leq \int_0^\pi \mid \frac{\log(Re^{i\pi})^2 iRe^{i\theta}}{R^2e^{wi\theta}+1} \mid d\theta$$

$$= \int_0^\pi \mid \frac{\left(\ln(R) + i\theta\right)^2 R}{R^2 e^{2i\theta} + 1} \mid d\theta$$

$$\leq \int_0^\pi \mid \frac{\ln^2(R) + 2i\theta \ln(R) - \theta^2}{R^2 e^{2i\theta} + 1} \mid R d\theta$$

$$\leq \frac{\left(\ln^2(R) + 2\pi \ln(R) + \pi^2\right) R}{R^2 - 1}$$

$\to 0$ as $R \to \infty$ since $\ln(R)/R \to 0$ as $R \to \infty$.

So $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0$.

$$\int_{\Gamma_\epsilon} f(z)dz = \int_0^\pi \frac{\log(\epsilon e^{i\theta})^2 i\epsilon e^{i\theta} d\theta}{\epsilon^2 e^{2i\theta} + 1}.$$

$$\mid \int_{\Gamma_\epsilon} f(z)dz \mid \leq \int_0^\pi \frac{\mid \ln(\epsilon) + i\pi \mid^2 \epsilon d\theta}{1 - \epsilon^2}$$

$$\leq \frac{(\ln(\epsilon)^2 + 2\pi \ln(\epsilon) + \pi^2)\, \epsilon\pi}{1 - \epsilon^2}$$

Now $\epsilon(\ln(\epsilon))^2 \to 0$ as $\epsilon \to 0$ by l'Hôpital, and $\epsilon \ln(\epsilon) \to 0$ as $\epsilon \to 0$ So $\int_{\Gamma_\epsilon} f(z)dz \to 0$ as $\epsilon \to 0$. So taking $\lim_{R\to\infty}$ and $\lim_{\epsilon\to 0}$, we find

$$\int_0^\infty \frac{(\ln(x)^2 dx}{1 + x^2} + \int_0^\infty \frac{\left(\ln(x) + i\pi\right)^2 dx}{1 + x^2}$$

$$= 2\pi i \mathrm{Res}_i f(z).$$

Now $f(z) = \frac{\log(z)^2}{(z+i)(z-i)}$ has simple pole at i so

$$\mathrm{Res}_i f(z) = \frac{(\log i)^2}{i + i} = (i\pi/2)^2 \frac{1}{2\pi} = -\frac{\pi^2}{8\pi}.$$

So

$$2\int_0^\infty \frac{\log(x)^2}{1 + x^2} - \pi^2 \int_0^\infty \frac{1}{1 + x^2} dx + 2i\pi \int_0^\infty \frac{\log(x) dx}{1 + x^2} = 2\pi i \left(-\pi^2/(8i)\right) = -\pi^3/4.$$

Taking real parts,

$$2\int_0^\infty \frac{(\log x)^2 xx}{1 + x^2} = -\pi^3/4 + \pi^2 \int_0^\infty \frac{dx}{1 + x^2}$$

$$= -\pi^3/4 + \pi^2 \tan^{-1}(x) \mid_0^\infty = -\pi^3/4 + \pi^3.2 = \pi^3/4.$$

So

$$\int_0^\infty \frac{(\log x)^2 dx}{1+x^2} = \pi^3/8.$$

7.

$$\int_0^\infty \frac{x^2 dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2 dx}{(1+x^2)^2}$$

Consider $\int_{\gamma_R} f(z)dz$ where γ_R is a closed semicircular contour with centre 0 and radius R. and $f(z) = \frac{z^2}{(1+z^2)^2}$. f has poles at $z = \pm i$, and the only pole inside the contour is $+i$. So for $R > 1$ $\int_{\gamma_R} f(z)dz = 2\pi i \text{Res}_{z=i} f(z)$

$$f(z) = \frac{z^2}{(1+z^2)^2}.$$

So for $R > 1$ $\int_{\gamma_R} f(z)dz = 2\pi i \text{Res}_{z=i} f(z)$.

$$f(z) = \frac{z^2}{(z+i)^2(z-i)^2} = \frac{h(z)}{(z-i)^2}$$

where $h(z) = \frac{z^2}{(z+i)^2}$ Thus $\text{Res}_{z=i} f(z) = h'(i)$

$$= \frac{2z}{(z+i)^2} - \frac{2z^3}{(z+i)^3} = \frac{2i}{(2i)^2} - \frac{2(-1)}{(2i)^3}$$

$$= \frac{1}{2\pi} - \frac{1}{4\pi} = \frac{1}{4\pi}.$$

So

$$\int_{\gamma_R} f(z)dz = \pi/2.$$

$$\mid \int_{z=Re^{i\theta}, 0\le\theta\le\pi} f(z)dz \mid \le \int_0^\pi \mid f(Re^{i\theta)} \mid Rd\theta$$

$$= \int_0^\pi \mid \frac{R^2}{(R^2 e^{2i\theta}+1)^2} \mid Rd\theta$$

$$\le \int \frac{R^3\pi}{(R^4 - 2R^2 - 1)^2} \to 0$$

as $R \to \infty$ Hence

$$\lim_{R\to\infty} \int_{z=Re^{i\theta}, 0\le\theta\le\pi} f(z)dz = 0$$

Hence

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(1 + x^2} = \pi/2$$

so

$$\int_{0}^{\infty} \frac{x^2 dx}{(1 + x^2)^2} = \pi/4.$$