8 Applications of Contour Integration

The improper integral of f over $\mathbf{R}$ is defined as

$$\lim_{R,S\to\infty}\int_{-S}^{R} f(x)dx.$$

The improper integral of f over $[0,\infty)$ is defined as

$$\lim_{R\to\infty}\int_{0}^{R} f(x)dx.$$

The principal value integral of f over $\mathbf{R}$ is

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx.$$

Example 8.1

$$\int_0^\infty \phi(x)\sin(mx)dx$$

or

$$\int_0^\infty \phi(x)\cos(mx)dx$$

where ϕ is a rational function.

Example 8.2

$$\int_0^\infty \frac{1}{(z^2+1)^2(z^2+4)}dx$$

Integrate $f(z)=\frac{1}{(z^2+1)^2(z^2+4)}$ around a semicircular contour Γ_R where the semicircle has centre 0 and radius R. The integrand has a simple pole at $2i$ and a double pole at i.

Hence

$$\int_{-R}^{R} f(x)dx+\int_{\Gamma_R} f(z)dz = 2\pi i\left(\mathrm{Res}(f(z),i)+\mathrm{Res}(f(z),2i)\right).$$

Now

$$\mathrm{Res}(f(z),i)=\frac{d}{dz}\left(\frac{1}{(z+i)^2(z^2+4)}\right)_i$$

$$=\left(\frac{-2z(z+i)^2-2(z^2+4)}{(z+i)^3(z^2+4)^2}\right)_i=-i/36,$$

and

$$\mathrm{Res}(f(z), 2i) = \frac{1}{(z^2+1)^2(2z)}_{2i} = -i/36$$

Also

$$|\int_{\Gamma_R} f(z)dz| \le \int_0^\pi \frac{1}{(R^2-1)^2(R^2-4)} Rd\theta$$
$$= O(R^{-5})$$

Also $|\int_{\Gamma_R} f(z)dz \mid \le \int_0^\pi \int \frac{1}{(R^2-1)^2(R^2-r)} Rd\theta = O(R^{-5})$ *and* $\int_{-R}^{R} = 2\int_0^R$ *so we have* $\pi/18$.

Example 8.3

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+x+1} dx$$

Integrate $f(z) = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+z+1}$ *around a semicircular contour* Γ_R *with center* 0 *and radius* R *in the upper half plane. By Cauchy,*

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma_R} f(z)dz = 2\pi i Res(f(z);\omega) = 2\pi i \frac{e^{i\omega}}{2\omega+1}.$$

This means

$$|\int_{\Gamma_R} f(z)dz \mid \le \int_0^\pi \frac{Re^{-R\sin\theta}}{R^2e^{2i\theta}+Re^{i\theta}+1} d\theta$$

which is of order R^{-1} *so it tends to* 0 *as* $R \to \infty$. *Taking the real part of the above equation, we have*

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+x+1} dx = \frac{2\pi}{\sqrt{3}} \cos(1/2) e^{-\sqrt{3}/2}.$$

Example 8.4 $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$ *Integrate* $f(z) = \frac{e^{iz}}{z}$ *around a contour which is a semicircle of radius* R *with a small semicircle around* 0 *removed. The function* f *is holomorphic except for a simple pole at* 0. *Because of the small semicircle,* f *is holomorphic inside the contour.*

By Cauchy,

$$\int_{-R}^{-\epsilon} f(x)dx - \int_{\Gamma_\epsilon} f(z)dz + \int_{\epsilon}^{R} f(x)dx + \int_{\Gamma_R} f(x)dx = 0.$$

The limit of $\int_{\Gamma_\epsilon} f(z)dz$ *as* $\epsilon \to 0$ *is* $i\pi$. *This means* $|\int_{\Gamma_R} f(z)dz| \leq 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta$ *which is of order* R^{-1}. *Hence letting* $R \to \infty$ *and* $\epsilon \to 0$ *we get*

$$i\pi = \lim_{R\to\infty,\epsilon\to 0} \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}dx = \lim_{R\to\infty,\epsilon\to 0} 2i \int_{\epsilon}^{R} \frac{\sin(x)}{x}dx.$$

So the answer is $\pi/2$.

8.1 Integrals involving functions with infinitely many poles

This treats integrals of the form $\int_{-\infty}^{\infty} \phi(x)e^{ix}dx$,

Example 8.5 $\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh(x)}dx$ *The function* $f(z) = \frac{e^{az}}{\cosh(z)}$ *has simple poles at* $z = \pi i(n + \frac{1}{2})$ *for integers* n. *This function is holomorphic inside and on a rectangular contour with vertices* $(-S,0),(R,0),(-S,\pi),(R,\pi)$, *except for a simple pole at* $i\pi/2$, *inside the contour, of residue* $-ie^{a\pi i/2}$. *By Cauchy's residue formula,*

$$\int_{-S}^{R} \frac{e^{ax}}{\cosh(x)}dx + \int_0^{\pi} \frac{e^{ax}}{\cosh(R+iy)}dy + \int_{R}^{-S} \frac{e^{a\pi i}\,{}^{ax}}{e}\cosh(x+i\pi)dx + \int_{-\pi}^{0} \frac{e^{a(-S+iy)}}{\cosh(-S+iy)}idy = 2\pi e^{a\pi i/2}.$$

Denote the second integral by I *and the fourth by* J. *Then*

$$|I| \leq \int_0^{\pi} \frac{2e^{aR}}{|e^{R+iy} + e^{-(R+iy)}|}dy \leq \int_0^{\pi} \frac{2e^{aR}}{|e^{R} - e^{-R}|}dy,$$

so $I \to 0$ *as* $R \to \infty$, *since* $a < 1$. *Likewise* $|J| \leq \int_0^{\pi} \frac{2e^{-aS}}{|e^{-S}-e^{S}|}dy$, *so* $J \to 0$ *as* $S \to \infty$, *since* $a > 1$.

Taking the limit as $R, S \to \infty$ *we get*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh(x)}dx = \frac{2\pi e^{a\pi i}/2}{1 + e^{a\pi i}} = \frac{2\pi}{e^{-a\pi i/2} + e^{a\pi i/2}} = \frac{\pi}{\cos \pi a/2}.$$