7. Cauchy's Residue Theorem

Definition 7.1. *If f is holomorphic on $D'(a;r)$ with a pole at a, the* residue *of f at a is the coefficient c_{-1} of $(z-a)^{-1}$ in the Laurent expansion of f about a (denoted $Res(f(z);a)$).*

Lemma 7.2. *Suppose f is holomorphic inside and on a positively oriented contour γ except at $a \subset \gamma$, where it has a pole. Then*

$$\int_\gamma f(z)dz = 2\pi i c_{-1}.$$

Proof: Choose r such that $\bar{D}(a;r) \subset I(\gamma)$. Then

$$\int_\gamma f(z)dz = \int_{\gamma(a;r)} f(z)dz$$

(by deformation theory)

$$= \int_{\gamma(a;r)} \sum_{n=-m}^{\infty} c_n (z-a)^n dz$$

$$= \sum_{n=-m}^{\infty} c_n \int_{\gamma(a;r)} (z-a)^n dz = 2\pi i c_{-1}.$$

Theorem 7.3 (Cauchy residue formula)**.** *Suppose f is holomorphic inside and on a positively oriented contour γ except for a finite number of poles at $a_1, \dots, a_m$ inside γ. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} Res(f(z);a_k).$$

Theorem 7.4 (Zero-pole theorem)**.** *Suppose f is holomorphic inside and on a positively oriented contour γ except for P poles and N zeros inside γ, and that f is nonzero on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P$$

(counting according to multiplicities).

Proof of Cauchy residue formula:

Proof. Let f_k be the principal part of the Laurent expansion of f around a_k. Then

$$g := f - \sum_{k=1}^{m} f_k$$

has only removable singularities at $a_1, \ldots, a_m$. Redefining it at the poles a_k, g is holomorphic inside and on γ. By Cauchy's theorem, $\int_\gamma g(z)dz = 0$. So by the previous Lemma $\int_\gamma f(z)dz = \sum_{k=1}^m \int_\gamma f_k(z)dz = 2\pi i \sum_{k=1}^m \mathrm{Res}(f(z); a_k)$. □

Proof of zero-pole theorem:

Proof. The function f'/f is holomorphic inside and on γ except at poles and zeros of f inside γ. If a is a zero of f of order m, then there is a function g which is holomorphic and nonzero in some $D(a; r)$ with $f(z) = (z-a)^m g(z)$ in $D(a; r)$. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

□

7.1. **Rouché's Theorem.**

Theorem 7.5 (Rouché)**.** *Suppose f and g are holomorphic inside and on a contour γ and suppose $|f(z)| > |g(z)|$ on γ. Then f and $f+g$ have the same number of zeros inside γ.*

Proof of Rouché: Let $t \in [0,1]$. Since $|f(z)| > |g(z)|$ on γ, $(f + tg)(z) \neq 0$ for any $z \in \gamma$. Assume WLOG γ positively oriented. Define

$$\phi(t) = \frac{1}{2\pi i}\int_\gamma \frac{(f' + tg')(z)}{(f+tg)(z)}dz.$$

By the zero-pole theorem, $\phi(t)$ equals the number of zeros of $f + tg$ inside γ. ϕ is integer-valued; if it is continuous, it must be a constant. So if $\phi(0) = \phi(1)$, then the number of zeros of f is equal to the number of zeros of $f + g$. To prove ϕ is continuous:

$$\phi(t) - \phi(s) = \frac{t-s}{2\pi i}\int_\gamma \frac{(g'f - f'g)(z)}{(f+tg)(z)(f+sg)(z)}dz$$

Since a continuous function on a compact set attains its maximum and its minimum, we can find positive numbers M and m such that for all $z \in \gamma$, $|\ (g'f - f'g)(z)\ | \leq M$, $|\ g(z)\ | \leq M$, $|\ (f+tg)(z)\ | \geq m$. Then $|\ (f+sg)z)\ | \geq |\ (f+tg)(z)\ | - |\ s-t\ | \, |g(z)| \geq m/2$ if $|s-t| \leq \frac{m}{2M}$. So for small enough $|s-t|$, by the estimation theorem, $|\phi(t) - \phi(s)| \leq \frac{|t-s|M}{\pi m}\mathrm{length}(\gamma)$. So ϕ is continuous at t.

7.2. **Zeros and residues.**

Lemma 7.6.

$$\mathrm{Res}(f(z);a) = \lim_{z\to a_n}(z-a)f(z).$$

Proof. In $D'(a;r)$, $f(z) = \sum_{n=-1}^{\infty} c_n(z-a)^n$ so $\lim_{z\to a}(z-a)f(a) = c_{-1}$. □

simple pole of type I: $g(z)(z-a)$ for g holomorphic in $D(a;r)$,

$$\mathrm{Res}(f(z);a) = g(a).$$

Simple pole of type II:

$$f(z) = \frac{h(z)}{k(z)}$$

h, k holomorphic in $D(a;r)$, $h(a) \neq 0$, $k(a) = 0$, $k'(a) \neq 0$

Then

$$\mathrm{Res}(f(z);a) = \frac{h(a)}{k'(a)}.$$

Multiple pole type I $g(z)(z-a)^{-m}$ for g holo. in $D(a;r)$:

$$\mathrm{Res}(f(z);a) = \frac{1}{(m-1)!}g^{(m-1)}(a).$$

Multiple pole type II: Compute c_{-1} in the Laurent expansion, or convert to type I.

Example 7.1. *How many zeros does $f(z) = 2 + z^2 - e^{iz}$ have in the upper half plane $z \in \mathbf{C}|\mathrm{Im}(\mathbf{z}) > \mathbf{0}\}$?*

Example 7.2. *Take $f(z) = 2 + z^2$, $g(z) = -e^{iz}$, and define a semi-circular contour Γ_R with diameter $2R$ and centre 0. On $[-R, R]$ (the part of the contour on the real axis), $|f(z)| \geq 2$, $|g(z)| = 1$. On the semicircular arc $z = Re^{i\theta}, 0 \leq \theta \leq \pi$,*

$$f(z) = 2 + R^2e^{2i\theta}.$$

$$|f(z)| \geq |\ |R^2e^{2i\theta}| - 2\ | = R^2 - 1$$

$$|g(z)| = e^{-R\sin(\theta)} \leq 1$$

(for $\sin(\theta) \geq 0$.) So again $|f(z)| \geq |g(z)|$, so by Rouché $f + g$ has the same number of zeros as f, which is 1.

Example 7.3.

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$\mathrm{Res}_{z=i}f(z) = \frac{1}{2i},\ \mathrm{Res}_{z=-i}f(z) = -\frac{1}{2i}$$

Example 7.4.

$$f(z) = \frac{1}{\sin(z)} = \frac{1}{z(1 - z^2/3! + \dots)}$$

$$\mathrm{Res}_{z=0}\frac{1}{\sin(z)} = 1.$$

Example 7.5.

$$f(z) = \frac{1}{(2-z)(z^2+4)} = -\frac{1}{(z-2)(z-2i)(z+2i)}$$

$$\mathrm{Res}_{z=2}f(z) = -\frac{1}{4+4} = -\frac{1}{8}.$$

$$\mathrm{Res}_{z=2i}f(z) = \frac{1}{(2-2i)(2i+2i)}$$

7.3. **Estimation of integrals.**

7.3.1. *I.* .

1. $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $|z_1 + \cdots + z_n| \leq \sum_{i=1}^{n} |z_i|$
3. $|z_1 + z_2| \geq ||z_1| - |z_2||$
4. $|z_1 + \cdots + z_n| \geq |z_1| - \sum_{j=2}^{n} |z_i|$

7.4. **Extra topic: Multifunctions.**

Example 7.6. *Examples of multifunctions:*

1. $\log(z)$ *defined for* $z \in \mathbf{C} \setminus \{\mathbf{0}\}$
2. $f(z) = z^\alpha$ $(\alpha \in \mathbf{C})$ *defined for* $z \in \mathbf{C} \setminus \{\mathbf{0}\}$
3. $f(z) = \log(p(z)/q(z))$ *defined except on the zeroes of* p *and* q *(where* p *and* q *are polynomials)*

Definition 7.7. *A multi-valued function* $f(z)$ *is the assignment to* z *of a set of complex numbers* $[w(z)]$*.*

Example 7.7. *Logarithm* $[\log(z)] = \{\log|z| + i\theta : \theta \in [arg(z)]\}$ *(in other words* $z = re^{i\theta}$*)*

Example 7.8. *Power* $[z^{1/n}] = \{|z|^{1/n}e^{2\pi i m/n} | m \in \mathbf{Z}\}$ *(there are* n *different values of* m*)*

Definition 7.8. *Let* $[w(z)]$ *be a multivalued function. A* branch point *of* w *is a point* $a \in \mathbf{C}$ *such that for sufficiently small circles* $\gamma(a, r)$ *around* a*, it is not possible to choose a continuous* $f(z)$ *of* $[w(z)]$ *defining a continuous function on* $\gamma(a, r)$*.*

Example 7.9. $\log(z)$ *is single valued on* $\mathbf{C} \setminus \{\mathbf{z} \in \mathbf{R} | \mathbf{z} \geq \mathbf{0}\}$*. Likewise* z^α *for any* $\alpha \in \mathbf{C}$ *since* z^α *is defined as* $\exp(\alpha \log(z))$*.*

7.5. **Logarithm.** $z = re^{i\theta}$

$$\log_k(r,\theta) = \log(r) + i(\theta + 2\pi k)$$

is a continuous function of r and θ ($r > 0$ and $\theta \in \mathbf{R}$) Note that $\log_k$ do not define a continuous choice of log on $\mathbf{C} \setminus \{\mathbf{0}\}$. As we move along a circle γ centred at 0, we pass from $\log_k$ to $\log_{k+1}$ as θ moves from 0 to 2π. To stop this from happening, we cut $\mathbf{C}$ along the negative real axis (or along any other half-line from 0 to ∞).

Example 7.10. *Powers*

$$P_k(r,\theta) = r^{1/n} e^{i\theta/n} e^{2\pi i k/n}$$

(for $r > 0$ and $\theta \in \mathbf{R}$) Then $P_k(r,\theta + 2\pi) = P_{k+1}(r,\theta)$ and $P_{n-1}(r,\theta + 2\pi)$ so there are really n branches of the function $z^{1/n}$, and they are permuted cyclically as we pass around a circle around 0 and θ goes from 0 to 2π.

Example 7.11.

$$f(z) = \sqrt{(z-1)(z+1)}$$

The set of branch points is $\{-1, 1\}$. Moving around a circular contour that winds around $+1$ or -1 with winding number 1, but not around both, will cause the function to pick up a factor of -1. To stop this from happening, we cut the plane along the line between 1 and -1. Along any contour in the cut plane, f is single valued.

Example 7.12.

$$f(z) = \log(\frac{z+i}{z-i})$$

Branch points $+i, -i, \infty$

Cut the plane from i to ∞ along the positive y axis.

Cut the plane from $-i$ to ∞ along the negative y axis.

Along the cut plane,

$$\frac{i}{2} \log\left(\frac{z+i}{z-i}\right)$$

is single valued. This is the definition of $\arctan(z)$ (the antiderivative of $\frac{1}{z^2+1}$): it is not single valued in the whole plane or upper half plane, only in the cut plane.