6 Laurent's Theorem

Theorem 6.1 *Let $A = \{z : R < |z - a| < S\}$ and suppose f is holomorphic on A. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

for $z \in A$ where

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a;r)} \frac{f(w)}{(w-a)^{n+1}} dw$$

for $R < r < S$. The c_n are unique.

Proof 6.1 *WLOG $a = 0$. Fix $z \in A$ and choose P and Q so that $R < P < |z| < Q < S$. Choose $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$; then*

$$f(z) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f(w)}{w-z} dw$$

(by Cauchy integral formula)

$$0 = \frac{1}{2\pi i} \int_{\tilde{\tilde{\gamma}}} \frac{f(w)}{w-z} dw$$

(by Cauchy's theorem) Hence

$$f(z) = \frac{1}{2\pi i} \int_{\gamma(0;Q)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \int_{\gamma(0;P)} \frac{f(w)}{w-z} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma(0;Q)} \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} f(w) dw - \frac{1}{2\pi i} \int_{\gamma(0;P)} \sum_{m=0}^{\infty} \frac{-w^m}{z^{m+1}} f(w) dw$$

using the binomial expansion. Use the Uniform Convergence Theorem to interchange summation and integration. This gives

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma(0;Q)} \left(\frac{f(w)}{w^{n+1}} dw \right) z^n$$

$$+ \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma(0;P)} f(w) w^m dw \right) z^{-m-1}.$$

Use the deformation theorem to replace $\gamma(0;Q)$ and $\gamma(0;P)$ by $\gamma(a;r)$ as in the statement.

Example 6.1

$$f(z) = \frac{1}{z(1-z)}$$

is holomorphic on A_1 and A_2, where

$$A_1 = \{z : 0 < |z| < 1\}$$

and

$$A_2 = \{z : |z| > 1\}.$$

On A_1,

$$f(z) = z^{-1} + (1-z)^{-1} = \sum_{n=-1}^{\infty} z^n$$

On A_2, we have

$$f(z) = z^{-1} - z^{-1}(1-z^{-1})^{-1} = \sum_{n=-\infty}^{-2} -z^n.$$

Example 6.2

$$f(z) = \frac{1}{z(1-z)^2}$$

is holomorphic on $0 < |z-1| < 1$. On this region it is equal to

$$\frac{1}{(z-1)^2}\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\Big(1-(z-1)+(z-1)^2-\dots\Big)$$

So

$$f(z) = \sum_{n=-2}^{\infty} (-1)^n (z-1)^n.$$

Example 6.3

$$\csc(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

on $0 < |z| < \pi$. Since

$$\sin(z) = z - \frac{z^3}{3!} + \dots$$

also

$$\begin{aligned}\csc(z) &= \frac{1}{z}\left(1 - \frac{(z^2)}{3!} + O(z^4))^{-1}\right) \\ &= \frac{1}{z}\left(1 + \frac{z^2}{3!} + \dots\right)\end{aligned}$$

Example 6.4

$$\begin{aligned}\cot(z) &= \left(1 - z^2/z! + \ldots\right)(1/z + z/3! + \ldots) \\ &= \frac{1}{z}\left(1 + z^2(-1/2 + 1/6) + O(z^4)\right).\end{aligned}$$

6.1 Singularities

Definition 6.2 *A point a is a* regular point *of f if f is holomorphic at a. It is a* singularity *of f if a is a limit point of regular points which is not itself regular.*

Definition 6.3 *f has* an isolated singularity *at a if f is holomorphic in a punctured disc $D(a;r) \setminus \{0\}$; if a is a singular point that does not satisfy this condition, it is called a* non-isolated singularity*.*

If f has an isolated singularity at a, f is holomorphic in the annulus $\{z : 0 < |z-a| < r\}$ and has a unique Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$. The singularity a is: *removable singularity* if $c_n = 0 \forall n < 0$; *pole of order* m if $c_{-m} \neq 0, c_n = 0 \forall n < -m$; *isolated essential singularity* if there does not exist m such that $c_n = 0\ \forall n < -m$.

In $D'(a,r)$, $f(z) = \sum_{n=-\infty}^{-1} c_n(z-a)^n + \sum_{n=0}^{\infty} c_n(z-a)^n$

Definition 6.4 *The* principal part *of the Laurent expansion is*

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n.$$

6.2 Zeros

Suppose f is holomorphic in $D(a;r)$ and $f(a) = 0$. Assume f is not identically zero in $D(a;r)$ (in other words f is not zero everywhere in $D(a;r)$).

Then by Taylor's theorem,

$$f(z) = \sum_{n=m}^{\infty} c_m(z-a)^m$$

for some $m \geq -1, c_m \neq 0$.

The *order* of zero of f at a is m if and only if $f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0$ but $f^{(m)}(a) \neq 0$.

Theorem 6.5 *Suppose f is holomorphic in $D(a;r)$. Then f has a zero of order m at a if and only if $\lim_{z\to a}(z-a)^{-m}f(z) = C$ for some constant $C \neq 0$.*

Theorem 6.6 *(Theorem 2) Suppose f is holomorphic on $D'(a;r)$. Then f has a pole of order m at a if and only if*

$$\lim_{z\to a}(z-a)^m f(z) = D$$

for a nonzero constant D.

Example 6.5 *$z\sin(z)$ has a zero of order 2 at $z=0$ and has zeros of order 1 at $z = n\pi$, $n \neq 0$.*

Proof of Theorem 2 $\Longrightarrow$ Suppose a is a pole of order m. For $z \in D(a;r)$, $z \neq a$,

$$f(z) = \sum_{n=-m}^{\infty} c_n(z-a)^n, c_{-m} \neq 0$$

In $D'(a;r)$,

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} c_{n-m}(z-a)^n.$$

The series on the right hand side defines a function continuous at $z = a$. Hence

$$\lim_{z\to a}(z-a)^n f(z) = c_{-m} \neq 0.$$

By Laurent's theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

where

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a;s)} \frac{f(w)}{(w-a)^{n+1}}dw$$

(for $0 < s < r$). We need $c_n = 0$ ($n < -m$) and $c_{-m} \neq 0$. Since $\lim_{z\to a}(z-a)^m f(z) = D \neq 0$, there is $\delta > 0$ such that

$$|(w-a)^m f(w) - D| < \epsilon$$

where $0 < |w-a| < \delta$. Take $0 < s < \min(\delta, r)$. Then if $|w-a| = s$, then $|(w-a)^m f(w)| \leq |D| + \epsilon Hence (w-a)^{-n-1} f(w)| \leq (|D| + \epsilon) s^{-m-n-1}$. So using the estimation theorem,

$$|c_n| \leq (|D| + \epsilon) s^{-n-m}.$$

If $n < -m$ then s^{-n-m} can be made arbitrarily small, but c_n is independent of s so $c_n = 0$. Hence

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n.$$

As in the proof of ($\Rightarrow$),

$$c_{-m} = \lim_{z \to a} (z-a)^m f(z) = D \neq 0.$$

6.3 Behaviour near an isolated singularity

Case 1: Removable singularity. If $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $D'(a;4)$. then $f(z) \to c_0$ as $z \to a$. Redefining $f(a)$ to c_0 we find f is holomorphic in $D(a;r)$.

Example 6.6 $f(z) = \frac{\sin(z)}{z}$

Case 2: Pole If f has a pole at a, then $|f(z)| \to \infty$ as $z \to a$.

Case 3: Essential singularity:

Casorati-Weierstrass theorem (not proved): If f has an isolated essential singularity at a, for any $w \in \mathbf{C}$ there exists $< a_n >$ such that $a_n \to a$ and $f(a_n) \to w$. In fact according to Picard's theorem, in any $D'(a,r)$, f assumes every complex value except possibly one. For example, $e^{1/z}$ has an essential singularity at 0; the value not assumed is 0.

Definition 6.7 *The extended complex plane* $\hat{\mathbf{C}}$ *is* $\mathbf{C} \cup \{\infty\}$ *(add an extra point at* ∞*).*

Define this by identifying

$$\hat{\mathbf{C}} = U \cup V / \sim$$

where $U = \mathbf{C}$, $V = \mathbf{C}$. On $\mathbf{C} \setminus \{\mathbf{0}\}$, identify U with V via $u \in U \setminus \{0\} \sim v \in V$ where $v = 1/u$. So as $u \to \infty$, $v \to 0$ and as $u \to 0$, $v \to \infty$. We can also write $\hat{\mathbf{C}}$ as $\{[z,w]\}/\sim$ where $(z,w) \sim (\lambda z, \lambda w)$ for any $\lambda \in \mathbf{C} \setminus \{\mathbf{0}\}$. Thus if $z \neq 0$, $(z,w) \sim (1, w/z)$ and if $w \neq 0$, then $(z,w) \sim (z/w, 1)$. These are in correspondence with the sets U and V.

Uniqueness

Theorem 6.8 *Suppose f is holomorphic on A with*

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n.$$

Suppose also

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

Then $b_n = c_n$.

Proof 6.2 *Assume $a = 0$. Choose r with $R < r < S$. Then*

$$2\pi i c_n = \int_{\gamma(0;r)} f(w) w^{-n-1} dw$$

$$= \int_{\gamma(0;r)} \sum_{k=-\infty}^{\infty} b_k w^{k-n-1} dw$$

$$= \int_{\gamma(0;r)} \sum_{k=0}^{\infty} b_k w^{k-n-1} dw = \int_{\gamma(0;r)} \sum_{m=-1}^{\infty} b_{-m} w^{-m-n-1} dw$$

Using the theorem on uniform convergence to interchange the sum with the integral,

$$2\pi i c_n = \sum_{k=-\infty}^{\infty} b_k \int_{\gamma(0,r)} w^{k-n-1} dw = 2\pi i b_n.$$

6.4 Meromorphic functions

Definition 6.9 *A $\mathbf{C}$-valued function which is holomorphic in an open set $G \subset \hat{\mathbf{C}}$ except possibly for poles is called* meromorphic *in G.*

Theorem 6.10 *If f is holomorphic on $\hat{\mathbf{C}}$, then f is constant.*

(Proof: Use Liouville's theorem)

Theorem 6.11 *If f is meromorphic on $\hat{\mathbf{C}}$ then f is a rational function $p(z)/q(z)$ for some polynomials p and q.*

Example 6.7 *1. $(z-a)^{-2}$ has a double pole at $z = 1$*

2. $(1-\cos(z))/z^2$ *is holomorphic except at* $z=0$. *At* $z=0$, $1-\cos(z) = z^2/2+\ldots$ *so the singularity is removable.*

3. $\frac{1}{\sin(z)} = \frac{1}{z-z^3/3!+\ldots} = \frac{1}{z(1-z^2/3!+\ldots)}$ *so there is a simple pole at* 0.

4. $$\cos(z)/\sin(z) = (1-z^2/2+\ldots)(1/z)(1+z^2/3!+O(z^4))$$

   $\cot(z)$ *has a simple pole at* $z=0$ *Similarly since* $\cot(z-k\pi) = \cot(z)$, $\cot(z)$ *has a simple pole at* $k\pi$.

5. $\sin(1/z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{-(2n+1)}}{(2n+1)!}$ *has an isolated essential singularity at* 0.

6. $\frac{1}{\sin(1/z)}$ *is singular when* $\sin(1/z)=0$, *in other words when* $1/z = k\pi$ *or* $z = 1/(k\pi)$ *for some integer* $k \neq 0$.

Remark 6.1 *If a is a limit point of the singularities of a function defined on a subset of* $\mathbf{C}$, *then f cannot be holomorphic on any punctured disk with centre at a, and cannot have a Laurent expansion about a. So a is not an isolated singularity or a regular point; it is a non-isolated essential singularity.*

Example 6.8

$$f(z) = \frac{1}{z^3\cos(1/z)}.$$

This has poles at $1/z = (2n+1)\pi/2$, *where* $\cos(1/z)$ *has zeros (in other words* $z = \frac{2}{(2n+1)\pi}$*). This expression tends to* 0 *as* $n \to \infty$ *so* 0 *is a limit point of the poles, or a non-isolated essential singularity. It follows that* f *does not have a Laurent expansion about* 0.

If f is meromorphic in an open subset G of $\tilde{C}$, then the set of poles of f has no limit point in G, and f can have at most finitely many poles in any closed subset of G.

6.5 *Behaviour of functions at* ∞

Zeros and poles of a function f at ∞ are studied by studying the function $\hat{f}(w) = f(1/w)$. f has a pole of order m at ∞ if and only if $\hat{f}$ has a pole of order m at 0.

f has a zero of order m at ∞ if and only if $\hat{f}$ has a zero of order m at 0.

Examples:

1.
$$f(z) = z^3$$
$$\hat{f}(w) = f(1/2) = w^{-3}$$
At $w = 0$, $\hat{f}$ has a pole of order 3.

2. $f(z) = \frac{1}{z^2} \sin \frac{1}{z}$. $\hat{f}(w) = w^2 \sin(w)$. $\hat{f}(w)$ has a zero of order 3 at $w = 0$.

3. $f(z) = z \sin(1/z)$
   $\hat{f}(w) = \sin(w)/w$ has a removable singularity at $w = 0$ if and only if $z = \infty$, in other words $w = 0$.

Example 6.9
$$f(z) = \frac{(z-1)^2 \cos(\pi z)}{(2z-1)(z^2+1)^5 \sin^3(\pi z)}$$
The denominator is zero at $z = 1/2$, $z = \pm i$ and $z = k \in \mathbf{Z}$.

$z = 1/2$:
$$(z^2+1)^5 \sin^3(\pi z) \neq 0$$
and $(z-1)^2 \neq 0$ but $\cos(\pi z) = 0$. So $z = 1/2$ is a removable singularity.

$z = \pm i$: $2z - 1 \neq 0$, $\sin^3(\pi z) \neq 0$, $z - 1 \neq 0$, $\cos(\pi z) \neq 0$. So $z^2 + 1 = (z+i)(z-i)$ and $z = \pm i$ are poles of $f(z)$ of order 5.

$z = k, k \neq 1$: $\cos(\pi z) \neq 0$, $z - 1 \neq 0$
$\sin(\pi z) = (\pi z - \pi k)(1 + higherorder)$
so $2z - 1 \neq 0, z^2 + 1 \neq 0$.
So $z = k$ is a pole of order 3, and $z = 1$ is a pole of order 1.

6.6 Meromorphic functions

Definition 6.12 *A $\mathbf{C}$-valued function which is holomorphic in an open set $G \subset \mathbf{C}$ except possibly for poles is called* meromorphic *in G.*

Theorem 6.13 *If f is holomorphic on $\tilde{\mathbf{C}}$, then f is constant.*

(Proof: Use Liouville's theorem)

Theorem 6.14 *If f is meromorphic on $\hat{\mathbf{C}}$ then f is a rational function $p(z)/q(z)$ for some polynomials p and q.*

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2. $(1-\cos(z))/z^2$ *is holomorphic except at* $z=0$. *At* $z=0$, $1-\cos(z) = z^2/2 + \ldots$ *so the singularity is removable.*

3. $\frac{1}{\sin(z)} = \frac{1}{z - z^3/3! + \ldots} = \frac{1}{z(1-z^2/3!+\ldots)}$ *so there is a simple pole at* 0.

4.
$$\cos(z)/\sin(z) = (1 - z^2/2 + \ldots)(1/z)(1 + z^2/3! + O(z^4))$$
$\cot(z)$ *has a simple pole at* $z=0$ *Similarly since* $\cot(z-k\pi) = \cot(z)$, $\cot(z)$ *has a simple pole at* $k\pi$.

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6. $\frac{1}{\sin(1/z)}$ *is singular when* $\sin(1/z) = 0$, *in other words when* $1/z = k\pi$ *or* $z = 1/(k\pi)$ *for some integer* $k \neq 0$.

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$$f(z) = \frac{1}{z^3 \cos(1/z)}.$$
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If f is meromorphic in an open subset G of $\tilde{C}$, then the set of poles of f has no limit point in G, and f can have at most finitely many poles in any closed subset of G.

Behaviour of functions at ∞

Zeros and poles of a function f at ∞ are studied by studying the function $\hat{f}(w) = f(1/w)$. f has a pole of order m at ∞ if and only if $\hat{f}$ has a pole of order m at 0.

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3. $f(z) = z \sin(1/z)$ $\hat{f}(w) = \sin(w)/w$ has a removable singularity at $w = 0$ if and only if $z = \infty$.

Example 6.12

$$f(z) = \frac{(z-a)^2 \cos(\pi z)}{(2z-1)(z^2+1)^5 \sin^3(\pi z)}$$

The denominator is zero at $z = 1/2$*,* $z = \pm i$ *and* $z = k \in \mathbf{Z}$*.*

$z = 1/2:$

$$(z^2+1)^5 \sin^3(\pi z) \neq 0$$

and $(z-1)^2 \neq 0$ but $\cos(\pi z) = 0$. So $z = 1/2$ is a removable singularity.

$z = \pm i$: $2z - 1 \neq 0$, $\sin^3(\pi z) \neq 0$, $z - 1 \neq 0$, $\cos(\pi z) \neq 0$. So $z^2 + 1 = (z+i)(z-i)$ and $z = \pm i$ are poles of $F(z)$ of order 5.

$z = k, k \neq 1$: $\cos(\pi z) \neq 0$, $z - 1 \neq 0$ $\sin(\pi z) = (\pi z - \pi k)(1 + \mathrm{h}igher order)$ so $2z - 1 \neq 0, z^2 + 1 \neq 0$. So $z = k$ is a pole of order 3, and $z = 1$ is a pole of order 1.

Example 6.13 *$f(z) = z \sin(z)$ has zeros at $z = n\pi$. So $1/f$ has poles at $z = k\pi$. At $z = 0$,* $\sin(z) = z - z^3/3! + \ldots = z(1 - z^2/3! + O(z^4))$ *so*

$$1/\sin(z) = 1/z(1 - z^2/3! + O(z^4))^{-1} = (1/z)(1 + (z^2/3! + \ldots) + (z^2/3! + \ldots)^2 + \ldots$$
$$= (1/z)(1 + z^2/3! + O(z^4))$$

So $1/(z\sin(z))$ has a double pole at $z = 0$. At $z = k\pi$, $k \neq 0$,

$$\sin(z) = (-1)^k \sin(z - k\pi)$$

and

$$z = k\pi + (z - k\pi)$$

So

$$\frac{1}{z} = \frac{1}{k\pi\,(1 + (z - k\pi)/k\pi)} = \frac{1}{k\pi} \sum_{n=0}^{\infty} (-1)^n (\frac{z - k\pi}{k\pi})^n.$$

So $1/(z\sin(z))$ has a simple pole at $z = k\pi$ when $k \neq 0$.

Example 6.14

$$f(z) = \cot(z) = \cos(z)/\sin(z)$$

Since $\sin(z) = 0$ *when* $z = k\pi$ *and* $\cos(k\pi) \neq 0$*, and* $\sin(z)$ *has a simple zero at* $z = k\pi$*,*

$$\sin(z) = (z - k\pi)(1 - (z - k\pi)^2/3! + \ldots)$$

we find $\cot(z)$ *has simple poles at these values.*

Non-isolated singularities are *always* essential. (Non-isolated singularity means there is no punctured disk $\{z : 0 \leq |z - a| < r\}$ where f is holomorphic.) If a is an isolated singularity (f is holomorphic in $D'(a; r)$ for some r), there is always a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

(i) Isolated essential singularity: Infinitely many nonzero c_j for $j < 0$
(ii) Pole: $f(z) = c_{-m}(z - a)^{-m} + c_{-m+1}(z - a)^{-m+1} + \ldots$ of order m.
(iii) Removable singularity: All $c_n = 0$ if $n < 0$.

$$\frac{1}{z^2 \sin(z)} = \frac{1}{z^2(z - z^3/3! + z^5/5! + \ldots)} = \frac{1}{z^3(1 - A)}$$

where $A = z^2/3! - z^4/5! + \ldots$

$$= \frac{1}{z^3}(1 + A + A^2 + \ldots)$$

Expand $1 + A + \ldots$ to order z^2 in z.