Proof[section]

5. Cauchy integral formula

Theorem 5.1. *Suppose f is holomorphic inside and on a positively oriented curve γ. Then if a is a point inside γ,*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw.$$

Proof. There exists a number r such that the disc $D(a,r)$ is contained in $I(\gamma)$. For any $\epsilon < r$,

$$\int_\gamma \frac{f(w)}{w-a}dw = \int_{\gamma(a;\epsilon)} \frac{f(w)}{w-a}dw$$

by the Deformation Theorem. So

$$\mid \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw - f(a) \mid = \mid \frac{1}{2\pi i}\int_{\gamma(a;\epsilon)} \frac{f(w)-f(a)}{w-a}dw \mid$$

$$= \mid \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})-f(a)}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}d\theta \mid$$

$$\leq \frac{1}{2\pi}(2\pi) \sup_{\theta\in[0,2\pi]} \mid f(a+\epsilon e^{i\theta}) - f(a) \mid$$

The right hand side tends to 0 as $\epsilon \to 0$. So the left hand side is 0. □

5.1. **Liouville's Theorem.**

Theorem 5.2. *If f is holomorphic on $\mathbf{C}$ and is bounded (in other words there exists M for which $|f(z)| < M$ for all z) then f is constant.*

Proof. Suppose $|f(w)| \leq M$ for all $w \in \mathbf{C}$. Fix a and b in $\mathbf{C}$. Take $R \geq 2max\{|a|,|b|\}$ so that $|w-a| \geq R/2$ and $|w-b| \geq R/2$ when $|w| = R$. By Cauchy's integral formula with $\gamma = \gamma(0;R)$,

$$f(a) - f(b) = \frac{1}{2\pi i}\int_\gamma f(w)\left(\frac{1}{w-a} - \frac{1}{w-b}\right)dw$$

$$\frac{a-b}{2\pi}\int_\gamma \frac{f(w)}{(w-a)(w-b)}dw.$$

So

$$\mid f(a) - f(b) \mid \leq \frac{1}{2\pi}2\pi RM\frac{|a-b|}{(R/2)^2}$$

by the Estimation Theorem. Since R is arbitrarily large, $LHS = 0$. □

5.2. **Fundamental theorem of algebra.**

Theorem 5.3. *Let p be a non-constant polynomial with constant coefficients. Then there exists $w \in \mathbf{C}$ such that $p(w) = 0$.*

Proof. Suppose not. Then $p(z) \neq 0$ for all z. Since $|p(z)| \to \infty$ as $|z| \to \infty$, there exists R such that $1/|p(z)| < 1$ for $|z| > R$. On $\bar{D}(0, R)$, $1/p(z)$ is continuous and hence bounded. Hence $1/p(z)$ is bounded on $\mathbf{C}$. It is also holomorphic, so constant by Liouville. □

5.3. **Cauchy's formula.**

Theorem 5.4. *Suppose f is holomorphic inside and on a positively oriented contour γ. Let a lie inside γ. Then $f^{(n)}(a)$ exists for $n = 1, 2, \ldots$ and*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw.$$

Corollary 5.5. *If f is holomorphic in an open set G, then f has derivatives of all orders in G.*

Proof. For $n = 0$ this is Cauchy's integral formula. We assume it is true for $n = k$ and prove it for $n = k + 1$. By deformation theorem, we may replace γ by $\gamma(a; 2r)$. Take $|h| < r$. By Cauchy's formula for $n = k$,

$$f^{(k)}(a+h) - f^{(k)}(a) = \frac{k!}{2\pi i} \int dw f(w) \left(\frac{1}{(w-a-h)^{k+1}} - \frac{1}{(w-a)^{k+1}} \right)$$

$$= \frac{(k+1)!}{2\pi i} \int_\gamma f(w) \left(\int_{[a,a+h]} (w-\zeta)^{-k-2} d\zeta \right) dw$$

(by Fundamental Theorem of Calculus for $f(z) = \frac{1}{z^{k+1}}$).

Define

$$F(h) = \frac{f^{(k)}(a+h) - f^{(k)}(a)}{h} = \frac{(k+1)!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{(k+2)}} dw$$

$$= \frac{(k+2)!}{2\pi i h} \int_\gamma f(w) \left(\int_{\zeta \in [a,a+h]} \Big(\int_{v \in [a,\zeta]} (w-v)^{-k-3} dv \Big) dz \right) dw$$

Since f is holomorphic (and hence continuous), it is bounded by some M on γ (since γ is compact). For $v \in [a, \zeta]$, $\zeta \in [a, h]$, $|w - v| \geq r$ for all $w \in \gamma$. Also $|zeta - a| \leq h$. Hence by the estimation theorem

$$|F(h)| \leq \frac{(k+2)!}{2\pi |h|} \frac{M|h|^2}{r^{k+3}} 4\pi r.$$

So $F(h) \to 0$ as $h \to 0$. □

Theorem 5.6 (Morera's Theorem)**.** *Suppose f is continuous in an open set G and $\int_\gamma f(z)dz = 0$ for all triangles γ in G. Then f is holomorphic on G.*

Proof. Let $a \in G$. Choose r so that $D(a;r) \subset G$. Since $D(a;r)$ is convex, the Antiderivative Theorem implies that there exists a holomorphic function F such that $F' = f$. Since F is holomorphic on $D(a,r)$, so is f. Since a is arbitrary, f is holomorphic on G. □

Example 5.1. *Use of Cauchy integral formula:*

1.

$$\int_{\gamma(i;1)} \frac{z^2}{z^2+1} dz$$

We need to write the integrand as $\frac{f(z)}{z-a}$ where f is holomorphic at a.

Take

$$f(z) = \frac{z^2}{z+i}$$

since $z^2 + 1 = (z+i)(z-i)$. Then

$$\int \gamma(i;1) \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

so

$$\frac{z^2}{(z+i)(z-i)} dz = 2\pi i \frac{z^2}{z+i} = -\pi$$

2.$\int_{\gamma(0;1)} \frac{e^z}{z^3} dz$

Rewrite as $\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$ which equals $2\pi f^{(n)}(a)$. We check that f is holomorphic inside and on a contour enclosing a. Take $f(z) = e^z, n =$ 2

$f'(z) = f''(z) = e^z$ so the integral is $\frac{2\pi i}{2!} f^{(2)}(0) = \pi i$

3.

$$\int_{\gamma(0;1)} \frac{\mathrm{Re}(z)}{z-1/2} dz$$

cannot be evaluated directly using the Cauchy integral formula since Re(z) is not a holomorphic function of z. But $\gamma(0;1)$ is the unit circle so $\mathrm{Re}(z) = (z+\bar{z})/2$ and if $|z| = 1$, $\bar{z} = z^{-1}$ so $\mathrm{Re}(z) = (z+z^{-1})/2$. So on $\gamma(0;1)$

$$\frac{\mathrm{Re}(z)}{z-1/2} = \frac{z^2+1}{2z(z-1/2)}$$

$$= \frac{z}{2(z-1/2)} + \frac{1}{z(2z-1)}$$

$$= \frac{1}{2}\frac{(z-1/2)+1/2}{z-1/2} + \frac{A}{z} + \frac{B}{2z-1}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{2z-1} + \frac{A(2z-1)+Bz}{z(2z-1)}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{2z-1} + \frac{(2A+B)z-A}{z(2z-1)}$$

Solving, $A = -1$ *and* $B = -2A = 2$.

So our expression is

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{2z-1} + \frac{(-1}{z} + \frac{2}{(2z-1)}$$

$$= \frac{1}{2} - \frac{1}{z} + \frac{5}{4(z-1/2)}$$

So by Cauchy's formula

$$\int_{\gamma(0;1)} \frac{\mathrm{Re}(z)}{z-1/2} dz = 2\pi i/4$$

5.4. ***Poisson integral formula.***

Theorem 5.7. *Suppose* f *is holomorphic inside and on* $\gamma(0;1)$. *Then*

$$f(re^{i\theta} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(e^{it})dt.$$

Proof. Fix $z = re^{i\theta}$. Apply Cauchy integral formula to $g(z) = f(z)\phi(z)$ where

$$\phi(w) = \frac{1-r^2}{1-w\bar{z}}.$$

Note that $\phi(z) = 1$. Then

$$f(z) = f(z)\phi(z)$$

$$= \frac{1-w^2}{2\pi i}\int_{\gamma(0;1)} \frac{f(w)}{(w-z)(1-w\bar{z})} dw$$

$$= \frac{1-r^2}{2\pi i}\int_0^{2\pi} \frac{f(e^{it})ie^{it}dt}{(e^{it}-re^{i\theta})(1-re^{it}e^{-i\theta})}$$

as required. □

5.5. **Power Series.**

Theorem 5.8. *(Taylor's Theorem) Suppose f is holomorphic on $D(a;R)$. Then there exist constants c_n so that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

and

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw = \frac{f^{(n)}(a)}{n!}$$

where γ is the circle $\gamma(a;r)$ for $0 < r < R$.

Proof. Fix $z \in D(a;R)$ and choose r so that $|z-a| < r < R$ Take $\gamma = \gamma(a;r)$. Then $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$ (by Cauchy integral formula). Since $|z-a| < |w-a|$ for all $w \in \gamma$,

$$\frac{1}{w-z} = \frac{1}{w-a}\frac{1}{1-\frac{z-a}{w-a}}$$

Now apply the binomial expansion:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} f(w)dw$$

Since γ is compact and f is continuous, f is bounded. So for some constant M,

$$\left| \frac{(z-a)^n}{(w-a)^{n+1}} f(w) \right| \leq \frac{M}{r}\frac{(z-a)}{r}^n := M_n.$$

Since $|z-a| < r$, $\sum_n M_n$ converges. So (by uniform convergence theorem) we may interchange summation and integration. So

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n.$$

Hence the theorem follows by Cauchy's formula for derivatives. □

Example 5.2. *Let f be holomorphic on $\mathbf{C}$. Prove that if there are $M > 0$ and $K > 0$, $0 < k \in \mathbf{Z}$ such that $|f(z)| \leq M|z|^k$ for $|z| \leq K$, then f is a polynomial of degree $\leq k$.*

Proof. By Taylor's theorem f has a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in any disk with center 0.

$$c_n = \frac{1}{2\pi i}\int_{\gamma(0;R)} f(z) z^{-n-1} dz.$$

So if $R \geq K$,

$$|c_n| \leq \frac{1}{2\pi} \sup\{|f(z)z^{-n-1}| : |z| = R\}$$

times the length of $\gamma(0; R)$. So this is

$$\leq \frac{1}{2\pi} M R^{k-n-1} (2\pi R).$$

Since R can be chosen arbitrarily large, we must have $c_n = 0$ for $n > k$. Thus f is a polynomial of degree $\leq k$. □

Proposition 5.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then*

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$$

where $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ If the radius of convergence of the power series for f is R_1 and that for g is R_2 then the radius of convergence of the power series for fg is at least the minimum of R_1 and R_2.

Example 5.3. *The power series for* $\exp(z)$ *has infinite radius of convergence (by the Ratio Test).*

5.6. **Zeros of holomorphic functions.**

Definition 5.9. *a is an isolated zero of f if there is $\epsilon > 0$ such that $D'(a, \epsilon)$ contains no zeros of f.*

Theorem 5.10. *(Identity Theorem) Suppose G is a region and f is holomorphic on G. If the set of zeros of f on G has a limit point in G then f is zero everywhere in G. (Equivalently the zeros of f are isolated unless f is zero everywhere.)*

Proof. Let $a \in G$ with $f(a) = 0$. By Taylor's theorem $(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ for $|z - a| \leq r$. Either all $c_n = 0$ (which implies $f = 0$ everywhere in $D(a; r)$) or there is a smallest $m > 0$ with $c_m \neq 0$. The series $\sum_{n=0}^{\infty} c_{n+m} (z - a)^n$ has radius of convergence at least r, and defines a function g which is continuous in $D(a; r)$ because a power series defines a holomorphic function inside its radius of convergence. Since $g(a) \neq 0$ and g is continuous at a, $g(z) \neq 0$ for z in some disk $D(a; \epsilon)$. In the punctured disk $D(a, \epsilon \setminus \{a\})$, $f(z) = (z - a)^m g(z)$ is never zero. So a is not a limit point of the set of zeros of f. Hence if $f(a) = 0$, either $f = 0$ in some disc, or a is not a limit point (in other words there is some disc where $f(z) \neq 0$ except for $z = a$). □

Sketch proof that if there is a limit point of the set of zeros in G, then $f = 0$: everywhere in G: Show that the set E of limit points is contained in the set $Z(f)$ of the set of zeros and both E and $G \setminus E$ are open (hence either $E = G$ or E is the empty set since G is connected).

(i) Suppose there is a limit point a for which $f(a) \neq 0$. Since it is a limit point, there are disks of radius $1/n$ containing points a_n with $f(a_n) = 0$ and $|a - a_n| < 1/n$. Since f is continuous this implies $f(a) = 0$ contrary to hypothesis. So

(ii) To show both E and $G \setminus E$ are open: Suppose $a \in E$. Then $f = 0$ in some disk D around a (by part 1). So this disk $D \subset E$. So E is open. To show $G \setminus E$ is open: Take $a \in G \setminus E$. Since a is not a limit point of $Z(f)$ there is a disc $D(a; r)$ in which f is never zero. By (i) $E \subset Z(f)$. Hence $D(a; r) \subset G \setminus E$.

5.7. **Maximum Modulus Theorem.**

Theorem 5.11. *Suppose f is holomorphic on $D(a, R)$ with $|f(z)| \leq |f(a)|$ for all $z \in D(a, R)$. Then f is constant.*

Theorem 5.12. *Suppose G is a bounded region, f is holomorphic in G and continuous on $\bar{G}$. Then $|f|$ attains its maximum on the boundary of G, in other words on $\partial G = \bar{G} \setminus G$.*

Proof of Theorem 5.11: Choose $0 < r < R$. By the Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_{\gamma(a;r} \frac{f(z)}{z - a} dz$$

$(\gamma(t) = re^{it})$

$$= \frac{1}{2\pi i} \int_0^{2\pi} (\frac{f(a + re^{it})}{rie^{it}} (re^{it}) dt$$

$$= \frac{1}{2\pi} \int f(a + re^{it}) dt.$$

Hence

$$|f(a)| \leq \frac{1}{2i} \int_0^{2\pi} |f(a + re^{it})| dt \leq |f(a)|.$$

(by hypothesis on f).

$$\int_0^{2\pi} \left[|f(a)| - f(a + re^{it}) \right] dt = 0.$$

(The integrand is continuous and ≥ 0, so it must be equal to 0.) This is true for all $r < R$. So $|f|$ is constant in $D(a; R)$. So f is also constant in $D(a; R)$. □

Proof of Theorem 5.12:

$\bar{G}$ is closed and bounded, so on $\bar{G}$, $|f|$ is bounded (as it is continuous) and attains its maximum value M at some point on $\bar{G}$. Assume $|f|$ does not attain M on the boundary ∂G. Then $|f(a)| = M$ for some $a \in G$. By part (a), f is constant on some disk $D(a;R) \subset G$. Hence f is constant in G, by the identity theorem. By continuity, f is constant on $\bar{G}$, so it attains its maximum on ∂G, contradicting the hypothesis. $\square$

Example 5.4 (Examples).

(1) *Is it possible to have a holomorphic function that is equal to* 0 *everywhere on the real axis?*

Answer: No, since the Identity Theorem says that if the set of zeros has a limit point then f is zero everywhere.

(A limit point is a point p for which any disc containing p, no matter how small, will contain some points z where $f(z) = 0$.)

(2) *Is it possible to have a holomorphic function which is equal to* 1 *when $z = \frac{1}{2n}$ and equal to -1 when $z = \frac{1}{2n+1}$?*

Answer: No, since the set of points $\{\frac{1}{2n}\}$ has a limit point (namely 0*) and $f = 1$ on those points. Likewise the set of points $\{\frac{1}{2n+1}\}$ has a limit point (namely* 0*) and $f = -1$ on those points. Looking at the points $\{\frac{1}{2n}\}$ we conclude $f = 1$ everywhere (by the identity theorem). Likewise looking at the points $\{\frac{1}{2n+1}\}$ we conclude $f = -1$ everywhere (by the identity theorem). This is a contradiction.*