2 Chapter 4: Cauchy's Theorem

Theorem 2.1 *Let γ be a positively oriented contour. Suppose $D(a;r) \subset I(\gamma)$. f is holomorphic inside and on γ except maybe at a. Then $\int_\gamma f(z)dz = \int_{\gamma(a;r)} f(z)dz$.*

For example, these hypotheses permit $f(z) = \frac{1}{z-a}$.

Proof 2.1 *Take a line ℓ through a which passes through no corner points of γ (points where one line segment joins another) and which is nowhere tangent to γ. Let z_1 and z_2 be points where ℓ meets the circle $|z-a| = r$. Also let w_1 and w_2 be points on $\gamma \cap \ell$ such that z_1 lies between a and z_1 and its absolute value is as small as possible. Form closed contours γ_1, γ_2 and then by Cauchy's theorem $\int_{\gamma_1} fdz = 0$. Similarly $\int_{\gamma_2} fdz = 0$. But then $\int_{\gamma_1} f + \int_{\gamma_2} f = \int_\gamma - \int_{\gamma(a;r)}$ since the integrals along line segments cancel.*

Definition 2.2 *The winding number of a closed path γ around a point w is*

$$n(\gamma;\omega) = \frac{1}{2\pi i}\int \frac{1}{z-w}dz.$$

Example 2.1 *Let $\gamma(t) = e^{it}$ and $w = 0$. Then $n(\gamma;0) = 1$. (The curve winds once around the origin.) But if instead $\gamma(t) = e^{2it}$ for $0 \leq t \leq 2\pi$, then $n(\gamma;0) = 2$. (The curve winds twice around the origin.)*

Theorem 2.3 (Cauchy III) *Suppose G is a region and f is holomorphic on G. For any closed path γ in G such that $n(\gamma;w) = 0$ for all $w \notin G$, $\int_\gamma f(z)dz = 0$.*

Theorem 2.4 (Cauchy's theorem) *Suppose f is holomorphic inside and on a contour γ. Then $\int_\gamma f(z)dz = 0$.*

Theorem 2.5 (Antiderivative theorem) *Suppose G is a convex region and f is holomorphic on G. Then there is F holomorphic on G such that $F' = f$.*

2.1 Logarithms

Theorem 2.6 *Suppose G is an open disc not containing 0. Then there exists a function $f = \log_G$ such that $e^{f(z)} = z \ \forall z \in G$ and $f(z) - f(a) = \int_\gamma \frac{1}{w} dw$, where γ is any path in G with endpoints a and z. f is uniquely determined up to $f \mapsto f + 2\pi i \mathbf{Z}$.*

Proof 2.2 *The Antiderivative Theorem implies there is a holomorphic function f such that $\frac{df}{dz} = \frac{1}{z}$ everywhere in G.*

$$\frac{d}{dz}\left(ze^{-f(z)}\right) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0.$$

So

$$ze^{-fz} = C$$

or

$$Ce^{f(z)} = z.$$

By adding a constant to f, we may assume $C = 1$. So

$$f(z) - f(a) = \int_\gamma \frac{dw}{w}$$

by the Antiderivative Theorem.

Theorem 2.7 (Jordan Curve Theorem) *Let γ be a contour. Then γ divides the complex plane into two components $I(\gamma)$ and $O(\gamma)$, where $I(\gamma)$ and $O(\gamma)$ are both connected, $I(\gamma)$ is bounded and $O(\gamma)$ is unbounded.*

Sketch proof of Cauchy's theorem: (This assumes a stronger condition on f which we shall eventually deduce from the hypothesis that f is holomorphic, rather than assuming it.)

Proof 2.3 *Recall* Green's theorem *from MATB42: Suppose γ is a contour bounding a region R, so interior points of R are on the left of γ. Suppose P and Q are real-valued functions and $P, Q, \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ are continuous in R. Then*

$$\int_\gamma P dx + Q dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy.$$

Now consider a $\mathbf{C}$*- valued function*

$$f(z) = u(x, y) + iv(x, y)$$

holomorphic in R. Assume also that u_x, u_y, v_x, v_y are continuous. (NOTE: Later we will prove the theorem without assuming the partial derivatives of u and v are continuous.) Then $\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)$. By Green's theorem this equals

$$\int_R (-v_x - u_y)\, dxdy + i\int_R (u_x - v_y)\, dxdy.$$

But by the Cauchy-Riemann equations, the integrands are zero, so $\int_\gamma f(z)dz = 0$.

Proof 2.4 Prove first for a triangle: *The Fundamental Theorem of Calculus implies that $\int_{\tilde{\gamma}} p(z)dz = 0$ for a polynomial p and a triangular contour $\tilde{\gamma}$. Near a point Z, approximate f by $p(z) = f(Z) + (z - Z)f'(Z)$. Replace $\int_\gamma f(z)dz$ by the sum of integrals around small triangles where $p(z)$ is a good approximation to $f(z)$. Let $[p, q, r]$ be the triangle with vertices p, q, r. Let $\gamma = [u, v, w]$, and let u', v', w' be the midpoints of $[v, w]$, $[w, u]$ and $[u, v]$ respectively. Define $\gamma^0 = [u', v', w']$, $\gamma^1 = [u, w', v']$, $\gamma^2 = [w, u', w']$, $\gamma^3 = [w, v', u']$. So $I = \int_\gamma f(z)dz$*

$$= \sum_{k=0}^{3} \int_\gamma^k f(z)dz.$$

For at least one k,

$$\mid \int_{\gamma^k} f(z)dz \mid \geq |I|/4.$$

Relabel this triangle as γ_1. Repeat this procedure with γ^1 in place of γ. We get a sequence of triangles such that

1. *$\gamma_0 = \gamma$*
2. *For all n, $\triangle_{n+1} \subset \triangle_n$ (we are assuming $\triangle_n$ is a closed triangle with γ_n as its boundary)*
3. *The length of γ_n is $2^{-n}L$ where L is the length of γ*
4. *$4^{-n}|I| \leq \mid \int_{\gamma_n} f(z)dz \mid$ for all $n \geq 0$.*

$\cap_{n=0}^{\infty} \triangle_n$ contains a point Z common to all the $\triangle_n$.

Fix $\epsilon > 0$. f is differentiable at Z so for some r,

$$\mid f(z) - f(Z) - (z - Z)f'(Z) \mid < \epsilon|z - Z| \tag{1}$$

for all $z \in D(Z;r)$. *Choose* $N, D(Z;r)$ *so that* $\triangle^N \subset D(Z;r)$.

$$|z - Z| \subset 2^{-N} L \tag{2}$$

for all $z \in \triangle_N$. *Hence*

$$\int_{z \in \gamma_N} |\ f(Z) + (z - Z) f'(Z)\ |\ dz = 0 \tag{3}$$

So by (1)-(3) and the Estimation theorem,

$$|\int_{\gamma_N} f(z)dz \ | \leq \epsilon(2^{-N})L \times \mathrm{length}(\gamma_N) = \epsilon(2^{-N}L)^2.$$

By item (4) in above list of properties of the sequence of triangles, $|I| \leq \epsilon L^2$. *Since* ϵ *is arbitrary,* $I = 0$.

2.2 Indefinite integral theorem

Theorem 2.8 *Let* f *be a continuous complex valued function on a convex region* G *such that* $\int_\gamma f(z)dz = 0$ *for any triangle* γ *in* G. *Let* a *be an arbitrary point of* G. *Then the function* F, *defined by*

$$F(z) = \int_{[a,z]} f(w)dw,$$

is holomorphic in G *with* $F' = f$.

Proof 2.5 *Fix* $z \in G$ *and* $D(z;r) \subset G$ *so that if* $|h| < r$ *then* $z + h \in G$. *Compute* $\lim_{h \to 0}(F(z+h) - F(z))/h$. *We will show this equals* $f(z)$. *For* $|h| < r$, $[a,z]$, $[z,z+h]$ *and* $[a,z+h]$ *all lie in* G *since* G *is convex. By hypothesis* $\int_\gamma f = 0$ *if* γ *is the triangle* $[a,z,z+h]$. *Hence*

$$F(z+h) - F(z) = \int_{[a,z+h]} f(w)dw - \int_{[a,z]} f(w)dw = \int_{[z,z+h]} f(w)dw.$$

Also

$$\int_{[z,z+h]} dw = h.$$

So

$$|\ \frac{F(z+h) - F(z)}{h} - f(z)\ | = \frac{1}{|h|}\ |\int_{[z,z+h]} [f(w) - f(z)]dw\ |$$

$$\leq \frac{1}{|h|} |h| \mathrm{sup}_{w \in [z,z+h]} [f(w) - f(z)]$$

which tends to 0 *as* $h \to 0$, *by continuity of* f *at* z.

2.3 Antiderivative Theorem

Theorem 2.9 *Let G be a convex region and let f be holomorphic on G. Then there exists F holomorphic on G such that $F' = f$.*

(Combine Cauchy's theorem for triangles with the indefinite integral theorem. By Cauchy's theorem, f satisfies the hypotheses for the indefinite integrals theorem.)

2.4 Cauchy theorem for convex region

Theorem 2.10 *Let G be a convex region, and f holomorphic on G. Then $\int_\gamma f(z)dz = 0$ for any closed path γ in G.*

Proof 2.6 *Combine antiderivative theorem with Fundamental Theorem of Calculus. By antiderivative theorem, $f = F'$. By FTC, $\int_\gamma F' = 0$.*

2.5 Cauchy's theorem

Theorem 2.11 *Suppose f is holomorphic inside and on a contour γ. Then $\int_\gamma f(z)dz = 0$.*

Proof 2.7 *First suppose γ is a polygon. Decompose γ into a union of triangles (see text for proof that this is possible). Hence $\int_\gamma f(z)dz = \sum_{k=1}^N \int_{\gamma_k} f(z)dz$ for triangles γ_k. Note that the integrals along the inserted segments cancel out.*

Let γ be any contour, and G an open set containing $\gamma^* \cup I(\gamma)$ on which f is holomorphic. Approximate γ by a polygonal contour. Cover γ^* with disks $D_k = D(\gamma(t_k); m)$ ($k = 0, \ldots, N$, $t_0 < \ldots < t_N$), with $\gamma(t_0) = \gamma(t_N)$. WLOG each γ_k is a line segment or a circular arc, and the line segments $[\gamma(t_k), \gamma(t_{k+1})]$ between $\gamma(t_k)$ and $\gamma(t_{k+1})$ join to form a polygonal contour $\tilde{\gamma}$ for which $\tilde{\gamma}^* \cup I(\tilde{\gamma}) \subset \cup_{k=0}^N D_k \cup I(\gamma)$ so it is in G. Hence $\int_{\tilde{\gamma}} f(z)dz = 0$. Also for all k $\gamma_k \cup (-\tilde{\gamma}_k)$ is a closed path in the convex region D_k. (Here the minus sign denotes the same curve with the opposite orientation.) So by Cauchy for convex sets, $\int_{\gamma_k} f(z)dz = \int_{\tilde{\gamma}_k} f(z)dz$. Hence

$$\int_\gamma f(z)dz = \sum_{k=0}^{N-1} \int_{\gamma_k} f(z)dz = \sum_{k=0}^{N-1} \int_{\tilde{\gamma}_k} f(z)dz = \int_{\tilde{\gamma}} f(z)dz = 0.$$

Definition 2.12 *(Positively oriented contour) A contour is positively oriented if, as t increases, $\gamma(t)$ moves counterclockwise around any point in $I(\gamma)$.*

Definition 2.13 (Simply connected) *A region C is simply connected if any closed path in C can be shrunk to a point continuously.*

Theorem 2.14 (Cauchy II) *Suppose f is holomorphic in a simply connected region G. Then $\int_\gamma f(z)dz = 0$ for every closed path γ in G.*

Example 2.2 *$f(z) = 1/z$ is holomorphic on $\mathbf{C} \setminus \{\mathbf{0}\}$, and $\gamma(t) = e^{it}$. We know that $\int_\gamma dz/z = 2\pi i \neq 0$ so $\mathbf{C} \setminus \{\mathbf{0}\}$ is not simply connected.*