MATC34 Chapter I: Complex-valued Functions

Definition 1.1 *A complex-valued function ia a mapping which assigns to each* $z \in \mathbf{C}$ *a unique complex number* $f(z) \in \mathbf{C}$*.*

Definition 1.2 *A complex-valued function* f *is* differentiable *if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. This limit, if it exists, is denoted $f'(z)$*.*

Definition 1.3 *Let* G *be an open set in* $\mathbf{C}$*. If* f *is differentiable at every* $z \in \mathbf{C}$*, then* f *is said to be* holomorphic *or* analytic *in* G*. The set of complex-valued functions holomorphic on* G *is denoted* $H(G)$*.*

Cauchy-Riemann equations

Theorem 1.4 *(Cauchy-Riemann): Suppose* $f = u + iv$ *is a complex-valued function (where* u *and* v *are real-valued functions). If* f *is differentiable at* z *then*

$$u_x = v_y, u_y = -v_x.$$

Proof 1.1

$$f'(z) = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \frac{v(x+h,y) - v(x,y)}{h}$$

$$= u_x + iv_x.$$

But also

$$f'(z) = \lim_{h \to 0} \frac{u(x,y+k) - u(x,y)}{h} + i \frac{v(x,y+k) - v(x,y)}{h}$$

where $h = ik$ *and* $k \in \mathbf{R}$*. This limit is* $-iu_y + v_y$*. Equating the real and imaginary parts we find*

$$u_x = v_y, u_y = -v_x.$$

Example 1.1 *If* f *is defined by* $f(z) = |z|$*, then* f *is not differentiable anywhere.*

Proof 1.2 *$u(x,y) = \sqrt{x^2+y^2}, v(x,y) = 0$ so $u_x = \frac{x}{\sqrt{x^2+y^2}}$ and $u_y = \frac{y}{\sqrt{x^2+y^2}}$ but $v_x = v_y = 0$.*

For $z = 0$, $\frac{f(h)-f(0)}{h} = \frac{|h|}{h}$. This limit is 1 if we take the limit as $h \to 0$ for $h \in \mathbf{R}^+$, but it is -1 if we take the limit as $h \to 0$ for for $h \in \mathbf{R}^-$. Hence the limit does not exists.

Remark 1.1 *The fact that a function satisfies the Cauchy-Riemann equations does not guarantee that it is differentiable. For example $f(z) = z^5/|z|^4$ satisfies the Cauchy-Riemann equations at $z = 0$ but it is not differentiable there.*

Properties of holomorphic functions

1. If f and g are holomorphic in a set $S \subset \mathbf{C}$, and $\lambda \in \mathbf{C}$ then λf, $f+g$ and fg are holomorphic in S.

2. Chain rule: If f is holomorphic in S and g is holomorphic in $f(S)$, then $g \circ f$ is holomorphic in S and $(g \circ f)'(z) = g'(f(z))f'(z)$.

3. If $f(z) \neq 0$ for all $z \in S$, then $\frac{1}{f}$ is holomorphic in S and

$$\left(\frac{1}{f}\right)'(z) = -\frac{f'(z)}{(f(z))^2}.$$

Definition 1.5 *A series of complex numbers*

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \dots$$

converges to a number T if the sequence of partial sums

$$S_N = \sum_{n=1}^{N} z_n$$

converges to T. We then write

$$\sum_{n=1}^{\infty} z_n = T.$$

Remark 1.2 *Let $\{a_n\}$ be a sequence of complex numbers. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges ('absolute convergence implies convergence').*

Remark 1.3 *The number* $\sum_{n=1}^{\infty} |a_n|$ *is a series with real nonnegative terms, to which tests for convergence (Ratio Test, Root Test, etc.) apply.*

Comparison test: *If* $|c_j| \leq M_j$ *for all* j*, and* $\sum_{j=0}^{\infty} M_j$ *converges, then so does* $\sum_{j=0}^{\infty} c_j$*.*

Ratio test: *Suppose* $\frac{|c_{j+1}|}{|c_j|} \to L$ *as* $j \to \infty$*. Then the series* $\sum_{j=1}^{\infty} c_j$ *converges if* $L < 1$ *and diverges if* $L > 1$*.*

Power series

Given a series $\sum_{n=0}^{\infty} c_n z^n$ for $c_n \in \mathbf{C}$, its radius of convergence is $R := \sup\{|z| : \sum_{n=0}^{\infty} |c_n z^n| \text{ converges }\}$. In this definition, it is possible that $R = \infty$.

Examples of holomorphic functions:

1. $f(z) = z$; $\frac{df}{dz} = 1$
2. any polynomial in z
3. any rational function $f(z) = \frac{p(z)}{q(z)}$ on the set where $q(z) \neq 0$, where p and q are polynomials
4. convergent power series in z

Theorem 1.6 *If* $\sum_{n=0}^{\infty} c_n z^n$ *has radius of convergence* R*, then we can use it to define a function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *for* $|z| < R$*. Then* f *is holomorphic at any* z *for which* $|z| < R$*, and its derivative is obtained by differentiating the series term by term:*

$$f'(z) = \sum_{n=0}^{\infty} n c_n z^{n-1}$$

for $|z| < R$*. These power series have the same radius of convergence.*

Theorem 1.7 *If* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *is a power series having nonzero radius of convergence, then* f *has derivatives of all orders at* 0 *and for* $n = 0, 1, 2, \ldots,$ $f^{(n)}(z) = n! c_n$*.*

Elementary functions

Examples of functions defined by power series.

1. Exponential function $f(z) = e^z$ defined by $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

   Properties:

(a) $\frac{d}{dz}e^z = e^z$

(b) e^z is holomorphic on the whole complex plane $\mathbf{C}$

(c) $e^z > 0$ if z is a real number

(d) $e^z \neq 0$ for any $z \in \mathbf{C}$

(e) If $z \in \mathbf{C}$ then $|e^z| = e^{\mathrm{Re}(z)}$ and if $t \in \mathbf{R}$ then $|e^{it}| = 1$.

2. Binomial expansion

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

This series has radius of convergence 1 (in other words this definition of $f(z)$ is valid for $|z| < 1$, not for $|z| > 1$).

3. Trigonometric functions

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh(z) = \cos(iz)$$

$$\sinh(z) = -i\sin(iz)$$

Euler's formula:

$$e^{iz} = \cos(z) + i\sin(z)$$

$$2\cos(z) = e^{iz} + e^{-iz}$$

$$2i\sin(z) = e^{iz} - e^{-iz}$$

$\cos(z+\alpha) = \cos(z)$ for all z if and only if $\alpha = 2k\pi$ for $k \in \mathbf{Z}$

$\sin(z+\alpha) = \sin(z)$ for all z if and only if $\alpha = 2k\pi$ for $k \in \mathbf{Z}$

$e^{z+\alpha} = e^z$ for all z if and only if $\alpha = 2\pi ik$ for $k \in \mathbf{Z}$

From these equations, $\cos(z) = 0$ iff $z = (k + \frac{1}{2})\pi$.

$\sin(z) = 0$ iff $z = k\pi$

$e^z = 1$ iff $z = 2\pi ik$

$e^z = -1$ iff $z = \pi i(2k+1)$