University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/9

Complex Variables

Instructor: Prof. L. Jeffrey *Office:* IC-474
Telephone: (416)287-7265
Email: jeffrey@utsc.utoronto.ca

Review of Complex Numbers

1.1 Basics

A complex number $z = x + iy$ corresponds to the point in the plane designated by the ordered set (x, y) of real numbers.

The number x is called the *real part* of z, denoted $\mathrm{Re}(z)$.

Likewise, the number y is called the *imaginary part* of z, denoted $\mathrm{Im}(z)$.

1.2 Multiplication

Multiplication of complex numbers is defined by specifying that $i^2 = -1$, and that real numbers multiply in the usual way.

1.3 Polar form of complex numbers

We may write a complex number as

$$z = re^{i\theta}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. *Euler's formula* specifies that $e^{i\theta} = \cos\theta + i\sin\theta$. The number θ is called the *argument* of z and the collection of possible arguments is denoted $[\arg(z)]$. When z is specified, its argument is not uniquely determined: if θ is one possible value of the argument of z, so are $\theta + 2\pi n$ for all $n = 0, \pm 1, \pm 2, \ldots$. One may choose a unique value for the argument by insisting that the argument θ take its value satisfying

$$-\pi < \theta \leq \pi;$$

the value of the argument satisfying this constraint is called the *principal value* of the argument and is denoted $Arg(z)$.
Fact: $[\arg(z_1 z_2)] = \{\theta_1 + \theta_2 \mid \theta_1 \in [\arg(z_1)],\ \theta_2 \in [\arg(z_2)]\}$.

Suppose $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. In order that $z_1 = z_2$, we need that $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi n$ for some $n = 0, \pm 1, \pm 2, \ldots$

1.4 Modulus

The modulus of z is $|z| = \sqrt{x^2 + y^2}$. Geometrically the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. Note that while the inequality $z_1 < z_2$ is meaningless unless both z_1 and z_2 are real, the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

1.5 Complex conjugate

The complex conjugate of a complex number

$$z = x + iy$$

is

$$\bar{z} = x - iy.$$

We have $|z|^2 = z\bar{z}$.

1.6 Elementary properties

If z and w are complex numbers, we have:
1. $\bar{\bar{z}} = z$
2. $2\mathrm{Re}(z) = z + \bar{z}$; $2\mathrm{Im}(z) = z - \bar{z}$
3. $|\bar{z}| = |z|$
4. $\overline{z + w} = \bar{z} + \bar{w}$; $\quad \overline{zw} = \bar{z}\bar{w}$.
5. Note that $|zw| = |z||w|$, but that in general $|z + w| \neq |z| + |w|$.

1.7 Inverse

The multiplicative inverse of a complex number is $z^{-1} = \bar{z}/|z|^2$.

If $z = re^{i\theta}$, then $z^{-1} = r^{-1}e^{-i\theta}$. (Geometrically, the modulus of z^{-1} is the reciprocal of the modulus of z, and the ray through z^{-1} is the reflection in the real axis of the ray through z.)

1.8. Powers and roots of complex numbers

These are best handled by expressing the complex number in polar coordinates: $z = re^{i\theta}$

For $m = 1, 2, \ldots$ we then have

$$z^m = r^m e^{im\theta}$$

and

$$z^{1/m} = r^{1/m} e^{i(\theta + 2\pi n)/m}$$

for all $n = 0, \pm 1, \pm 2, \ldots$. All possible values are obtained by taking $n = 0, 1, \ldots, m-1$; there are m possible roots.
Example: Find the cube roots of $z = -2$.
Solution: $-2 = 2e^{i\pi}$ so $(-2)^{1/3} = 2^{1/3} e^{i(\pi + 2\pi n)/3}$
$= 2^{1/3}\omega$ where $\omega = e^{i\pi/3}, e^{i\pi}, e^{5i\pi/3}$.

1.9 Multiplication in polar coordinates

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Example. $z_1 = 1 + i, z_2 = -1 + i$
$z_1 = \sqrt{2} e^{i\pi/4}, \quad z_2 = \sqrt{2} e^{3i\pi/4}$
$\Rightarrow z_1 z_2 = 2e^{i\pi} = -2.$