University of Toronto at Scarborough
Division of Mathematical Sciences

Midterm Test
MATC34H
Complex Variables

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Duration: 110 minutes

1. (a) Is the function
$$f(z) = z|z|^2$$
differentiable for any points z in the complex plane? If so, for which ones? What is the derivative at these points?
Solution:
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2\overline{z+h} - z^2\bar{z}}{h}$$
$$= \lim_{h \to 0} \frac{2z\bar{z}h + z^2\bar{h}}{h}$$
This gives $2|z|^2 + z^2\frac{\bar{h}}{h}$ The latter limit is not well defined because if $h = re^{i\theta}$ it gives $e^{-2i\theta}$ which depends on the angle θ. So the derivative only exists at $z = 0$

   (b) Let $g(z) = \bar{z}$. Compute the limit
$$\lim_{z \to 0} g(z+2) \ .$$
Solution: This is $\lim_{z \to 0} \overline{z+2} = 2$

2. (a) Compute the values of the following integrals around the circle $\gamma(1;1)$, with centre 1 and radius 1, and with counterclockwise orientation:
$\displaystyle\int_{\gamma(1;1)} f(z)dz$, where $f(z) = \dfrac{1}{(z-1)(z-i)}$.
Solution: Partial fractions
$$\frac{1}{(z-1)(z-i)} = \frac{A}{z-1} + \frac{B}{z-i} = \frac{A(z-i) + B(z-1)}{(z-1)(z-i)}$$
This gives $A + B = 0$ and $-iA - B = 1$ so $A = \frac{1}{1-i} = (1+i)/2$.
The circle contains 1 and does not contain i so we get $2\pi i B$ by the Deformation Theorem.

(b) Let γ be the square with vertices $2-2i$, $2+2i$, $-2+2i$, $-2-2i$ traversed in the counterclockwise direction. Compute
$\int_\gamma f(z)dz$ where $f(z) = \dfrac{1}{(z-i)(z-3i)}$.
Solution: The square contains i but does not contain $3i$. Using partial fractions

$$\frac{1}{(z-i)(z-3i)} = \frac{A}{z-i} + \frac{B}{z-3i}$$

$$= \frac{A(z-3i)+B(z-i)}{(z-i)(z-3i)}$$

so $A=-B$ and $-3iA+iA=1$ or $i(-2A)=1$ or $A=i/2$. So by the Deformation Theorem the integral is $2\pi i(i/2)=-1$.

3. Can there be a function F with
$$\frac{dF}{dz} = \frac{1}{z}$$
for all z in the punctured disc $D'(0,1)$ with centre 0 and radius 1? Solution: No. Such a function would have to equal $\log(z)$ but $\log(z)$ is a multivalued function on the punctured disc. $\log(re^{i\theta}) = \ln(r) + i\theta$ but θ is not well defined on the punctured disc (since $e^{i\theta} = e^{i(\theta+2\pi)}$).

4. (a) State the Cauchy-Riemann equations.
Solution: $u_x = v_y, v_x = -u_y$

(b) Let f be a holomorphic function,

$$f(x+iy) = u(x,y) + iv(x,y)$$

for which $u(x,y) = v(x,y)$ for all x and y. Show that f is constant.
Solution: Same solution as midterm 01 question about a holomorphic function which is real everywhere (instead of its being real everywhere it is a real multiple of $1+i$ everywhere)