# MATA33 WINTER 2009
# Test #2 SOLUTIONS

✱ **Print letters for the Multiple Choice questions in these boxes.**

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| b | a | c | c | c | c |

BLANK boxes earn a score of 0

**Do not write anything in the boxes below.**

3 points iff ALL info on page 1 is completed. ANYTHING MISSING → 0 pts

| Info | Part A |
|---|---|
| ☺ | |
| 3 | 24 |

**Part B**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| | | | | | | |
| 14 | 12 | 7 | 8 | 8 | 7 | 17 |

| Total |
|---|
| |
| 100 |

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TEST #2 SOLUTIONS

**Part A - Multiple Choice Questions** Print the letter of the answer you think is correct in the mark box at the top of page 2. Each right answer earns 4 points. Each blank mark box or wrong answer earns 0 points. A small space is provided for your rough work. (✱)

1. If $f(x, y) = 5x^2 ln(x^2 - 3y)$ then $f_x(2, 1)$ equals

   (a) 20 (b) 80 (c) 81 (d) 101 (e) none of (a) - (d)

$$f_x(x,y) = 10x \ln(x^2-3y) + \frac{5x^2(2x)}{x^2-3y}$$

$$f_x(2,1) = 20 \ln(1) + \frac{80}{1} = \boxed{80}$$

2. The joint demand functions for products $A$ and $B$ are $\alpha(x, y) = 80 - 2x + e^{-y} - y^2$ and $\beta(x, y) = 140 + (4x)^{-1} - 7y$ respectively, and $x$ and $y$ are the unit prices for $A$ and $B$, respectively. We may then conclude that $A$ and $B$ are

   (a) complementary (b) competitive (c) both (a) and (b) (d) neither (a) nor (b)

$$\alpha_y = -e^{-y} - 2y < 0$$

$$\beta_x = -\frac{1}{x} - 7 < 0$$

3. Exactly how many of the following properties are equivalent to the statement, "For a given $n > 1$, the $n \times n$ matrix $A$ is invertible".

   ✓ (i) $det(A) \neq 0$

   ✓ (ii) The reduced matrix of $A$ is the $n \times n$ identity

   ✗ (iii) $A + A^T$ is invertible

   ✗ (iv) There is a matrix $C$ such that $AC = CA$

   ✓ (v) The matrix $A^3$ is invertible

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

4. The domain of the function $f(x,y) = \dfrac{(1-x^2-y^2)^{1/2}}{x^2+y^2}$ is the set of points that

(a) lie on or inside the unit circle
(b) lie strictly inside the unit circle, except $(0,0)$
(c) lie on or inside the unit circle, except $(0,0)$
(d) lie on or outside the unit circle
(e) lie strictly outside the unit circle

$x^2+y^2 \le 1$
$x^2+y^2 \ne 0$ .

5. If $w = \sqrt[3]{rs}\, e^{5+r}$ then $\dfrac{\partial w}{\partial s}$ equals

$w = r^{1/3} s^{1/3} e^{5+r}$

(a) $\dfrac{w}{s}$ (b) $\dfrac{3w}{s}$ (c) $\dfrac{w}{3s}$ (d) $\dfrac{sw}{3}$ (e) none of (a) - (d)

$$\frac{\partial w}{\partial s} = r^{1/3}\frac{s^{-2/3}}{3} e^{5+r}$$
$$= \frac{r^{1/3} s^{1/3} e^{5+r}}{3s}$$
$$= \frac{w}{3s}$$

6. The equation of the level curve of $f(x,y) = \dfrac{6xy}{x^2+2} + 1$ that passes through $(1,3)$ is

(a) $y = \dfrac{4}{3}\left(\dfrac{x^2+2}{x}\right)$ (b) $y = \dfrac{x^2+2}{2x}$ (c) $y = \dfrac{x^2+2}{x}$ (d) $y = \dfrac{x}{x^2+2}$
(e) none of (a) - (d)

$$f(1,3) = \frac{18}{3} + 1 = 7$$
$$\frac{6xy}{x^2+2} + 1 = 7 \Rightarrow xy = x^2+2$$
$$\boxed{\therefore\ y = \frac{x^2+2}{x}}$$

**(Check that your Multiple Choice answers in the mark boxes on page 2)**

**Part B - Full Solution Problem Solving** Full points are awarded for solutions that are numerically correct and sufficiently display concepts and methods in the curriculum of MATA33.

1. Let $x$ be a real number and let $M = \begin{bmatrix} 2 & x & 6 \\ 2 & 7 & x \\ 2 & 7 & 7 \end{bmatrix}$

(a) Find and simplify $det(M)$ Expand along row 1 [6 points]

$$\det(M) = 2\begin{vmatrix} 7 & x \\ 7 & 7 \end{vmatrix} - x\begin{vmatrix} 2 & x \\ 2 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 7 \\ 2 & 7 \end{vmatrix}$$

$$= 2(49 - 7x) - x(14 - 2x) + 0$$

$$= 98 - 14x - 14x + 2x^2$$

$$= 2(x^2 - 14x + 49) = \boxed{2(x-7)^2}$$

(b) Find $M^{-1}$ when $x = 6$ [8 points]

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 1 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/2 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\boxed{\therefore M^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

2. In all of this question let $I(x, y) = x^2 - 4x - y + 8$. $I(x, y)$ represents the annual income in thousands of dollars obtained by holding $x$ units of stock $X$ and $y$ units of stock $Y$ in your investment portfolio for one year. Assume $x$ and $y$ are real and $\geq 0$.

(a) Let $A$ be the set of points $(x, y)$ for which the annual income $I(x, y)$ is at least \$ 3,000 Draw a neat, labeled diagram of the set $A$. [7 points]

We want $I(x,y) \geqslant 3$ and

$A = \{(x,y) \mid I(x,y) \geqslant 3\}$, $x, y \geqslant 0$

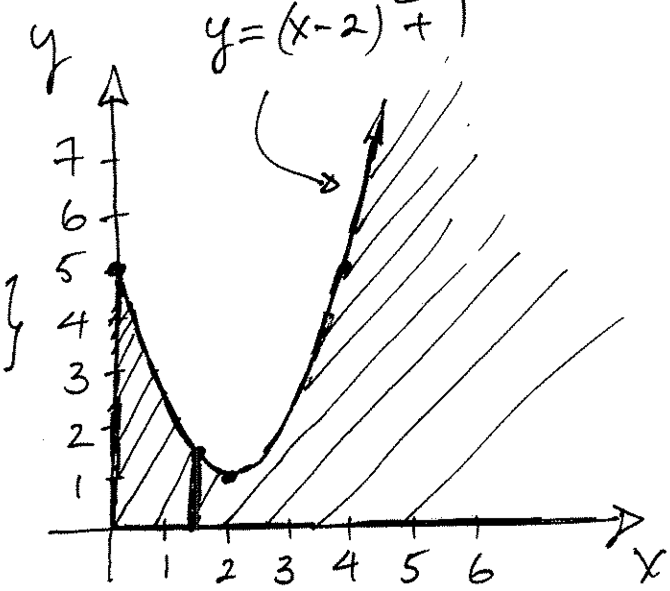
$x^2 - 4x - y + 8 \geqslant 3$

$x^2 - 4x + 4 - y + 1 \geqslant 0$

$(x-2)^2 + 1 \geqslant y$

$A = \{(x,y) \mid x, y \geqslant 0,\ y \leq (x-2)^2 + 1\}$

A is shaded ////




(b) Describe mathematically the set of points $(x, y)$ in $A$ such that $I_x(x, y) = I_y(x, y)$. [5 points]

Consider $(x,y) \in A$ such that $I_x(x,y) = I_y(x,y)$

We want $2x - 4 = -1$ so $x = \frac{3}{2}$

$I(\frac{3}{2}, y) \geq 3 \Leftrightarrow 0 \leq y \leq (\frac{3}{2} - 2)^2 + 1 = \frac{5}{4}$

$\therefore$ the set of points we want

is $E = \{(x,y) \mid x = \frac{3}{2},\ 0 \leq y \leq \frac{5}{4}\}$

(Vertical line in A above is E)

$$\overbrace{2z^2 + 2x^2z^3}^{(*)} = xy$$

3. Assume the equation $2z^2 + 2x^2z^3 = xy$ defines $z$ implicitly as a function of independent variables $x$ and $y$. Find the value of $z_x$ at the point $(x, y, z)$ where $y = 4$ and $z = 1$.

[7 points]

Partial differentiate w.r.t. $x$ :

$$4zz_x + 4xz^3 + 6x^2z^2z_x = y$$

$$z_x(4z + 6x^2z^2) = y - 4xz^3$$

$$z_x = \frac{y - 4xz^3}{4z + 6x^2z^2} \qquad z_x(1,4,1) = \frac{0}{10}$$

When $y=4$, $z=1$ in $(*)$, $\qquad = 0$

$$2 + 2x^2 = 4x \qquad \boxed{\text{Answer} = 0}$$

$$2x^2 - 4x + 2 = 0$$

$$2(x-1)^2 = 0 \to x = 1$$

4. Let $f(x, y) = e^{xy}$ and let $F(x, y) = f_y(x, y) - f_{xy}(x, y)$ Find the function $y = g(x)$ such that $F(x, g(x)) = 0$ for all $x$ in the domain of $g$.

[8 points]

$$f_y(x,y) = e^{xy} \cdot x$$

$$f_{xy}(x,y) = f_{yx}(x,y) = e^{xy}xy + e^{xy}$$

$$F(x,y) = e^{xy}x - e^{xy}xy - e^{xy}$$

$$= e^{xy}(x - xy - 1)$$

$$F(x,y) = 0 \Leftrightarrow x - xy - 1 = 0$$

$$\Leftrightarrow y = \frac{x-1}{x}$$

$$\boxed{\therefore\ g(x) = \frac{x-1}{x},\ x \neq 0}$$

5. Let $c$ be a real constant and let $(*)$ represent the system of two linear equations:

$$x - 2y = cx + 5 \qquad (1-c)x - 2y = 5$$
$$x - y = cy - 7 \qquad x + (-1-c)y = -7$$

Use Cramer's rule to solve $(*)$. [8 points]

$$\det\begin{pmatrix} 1-c & -2 \\ 1 & -1-c \end{pmatrix} = -(1-c)(1+c) + 2$$
$$= -1 + c^2 + 2 = c^2 + 1$$

$$x = \frac{\det\begin{pmatrix} 5 & -2 \\ -7 & -1-c \end{pmatrix}}{c^2+1} = \frac{-5-5c-14}{c^2+1} = \frac{-5c-19}{c^2+1}$$

$$y = \frac{\det\begin{pmatrix} 1-c & 5 \\ 1 & -7 \end{pmatrix}}{c^2+1} = \frac{-7+7c-5}{c^2+1} = \frac{7c-12}{c^2+1}$$

6. Assume $B$ is an $n \times n$ matrix, $n > 1$, and $det(B) = 4$.

(a) Find $n$ such that $det(-5B) = 100$ [3 points]

We have $(-5)^n \det(B) = 100$

$\Rightarrow (-5)^n = 25$ so $\boxed{n = 2}$

(b) $det\left(\left(\frac{B^2}{2}\right)^{-1}\right) = \dfrac{1}{\det\left(\frac{B^2}{2}\right)} = \dfrac{1}{\left(\frac{1}{2}\right)^n 4^2}$ [4 points]

$$= \frac{1}{\left(\frac{4^2}{2^n}\right)} = \frac{2^n}{4^2} = \boxed{2^{n-4}}$$

7. In all of this question $C = \frac{xy}{5x+3y}$ is the manufacturing cost function (in millions of dollars) where $x, y > 0$ are the number of hundreds of units of two products, $P$ and $Q$, respectively.

(a) Find and simplify the marginal cost functions. [4 + 4 points]

$$C_x = \frac{y(5x+3y) - 5xy}{(5x+3y)^2} = \frac{3y^2}{(5x+3y)^2}$$

$$C_y = \frac{x(5x+3y) - 3xy}{(5x+3y)^2} = \frac{5x^2}{(5x+3y)^2}$$

(b) Find and interpret the meaning of $\frac{\partial C}{\partial x}(3, 1)$ [3 points]

$$\frac{\partial C}{\partial x}(3,1) = C_x(3,1) = \frac{3}{(18)^2} = \frac{1}{108}$$

Interpretation $\frac{1}{108} \approx C(4,1) - C(3,1)$

$\underline{\underline{\text{ie}}}$ Cost to manufacture the $4^{th}$ batch of 100 units of P while making 100 units of Q is $\approx 1/108$

(c) Find the number of units of $P$ and $Q$ manufactured under the assumptions that (i) the total number manufactured is 1,000 units and (ii) the marginal cost functions are equal. Round your answers to the nearest unit. [6 points]

(i) $\Rightarrow$ $x + y = 10$ ($x, y$ are in units of 100's)

(ii) $C_x = C_y \Rightarrow \frac{3y^2}{(5x+3y)^2} = \frac{5x^2}{(5x+3y)^2}$

$\therefore$ $y = \sqrt{\frac{5}{3}}\, x$ as $x, y > 0$

By (i), $x\left(1 + \sqrt{\frac{5}{3}}\right) = 10$

$\therefore$ $x = \frac{10}{1+\sqrt{\frac{5}{3}}} \approx 4.3649$

$y \approx 5.635$

$\boxed{\therefore \text{ 436 of P and 564 of Q.}}$

(extra answer space for Part (c) is on the next page)

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