

Solutions Assignment 9 MATA33

① Problems from Section 17.6

#2 $f(x,y) = x^2 + 4y^2 - 6x + 16y$

$$f_x(x,y) = 2x - 6 = 0 \rightarrow x = 3$$

$$f_y(x,y) = 8y + 16 = 0 \rightarrow y = -2$$

The (only) CP
of f is
 $(3, -2)$



#6 $f(x,y,z,w) = x^2 + y^2 + z^2 + w(x+y+z-3)$

$$f_x(x,y,z,w) = 2x + w = 0 \rightarrow w = -2x \quad ①$$

$$f_y(x,y,z,w) = 2y + w = 0 \rightarrow w = -2y \quad ②$$

$$f_z(x,y,z,w) = 2z + w = 0 \rightarrow w = -2z \quad ③$$

$$f_w(x,y,z,w) = x + y + z - 3 = 0 \quad ④ \quad \leftarrow$$

① + ② $\rightarrow y = -x$ ① + ③ $\rightarrow z = -x$. Sub in ④ to get

$$x + x + x - 3 = 0 \rightarrow x = 1 \rightarrow w = -2$$

\therefore the (only) CP of f is $(1, -1, 1, -2)$



#8 Examine the function $f(x,y) = -2x^2 + 8x - 3y^2 + 24y + 7$ for relative (i.e. local) extrema.

2nd DT:

Solution: I obtain CP(s) of f { $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

$$f_x = -4x + 8 = 0 \rightarrow x = 2 \quad \left\{ \begin{array}{l} \\ \\ \end{array} \right. \quad = (-4)(-6) - 0 \\ = 24$$

$$f_y = -6y + 24 = 0 \rightarrow y = 4 \quad \therefore D(2,4) > 0 \quad f_{xx}(2,4) = -4 < 0$$

\therefore the only CP is $(2,4)$ $\therefore f$ has rel. max at $(2,4)$ of value $f(2,4) = 63$



(2)

#10 Examine $f(x,y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$ for local extrema.

$$\text{Solution: } f_x(x,y) = 4x + 3y - 10 = 0 \quad ①$$

$$f_y(x,y) = 3y + 3x - 9 = 0 \quad ②$$

$① - ② \rightarrow x - 1 = 0$ so $x = 1$. Sub in ① $\rightarrow y = 2$
 \therefore the CP of f is $(1, 2)$

$$\begin{aligned} \text{2nd DT: } D(x,y) &= f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2 \\ &= (4)(3) - 3^2 = 3 \end{aligned}$$

$$\therefore D(1,2) = 3 > 0 \quad f_{xx}(1,2) = 4 > 0$$

$\therefore f$ has a relative minimum at $(1,2)$ of value $f(1,2) = -12$



#12 Examine $f(x,y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$ for local extrema.

$$\text{Solution: } f_x(x,y) = x^2 - 2 - 2y = 0 \quad ①$$

$$f_y(x,y) = 2y + 2 - 2x = 0 \quad ②$$

① $\rightarrow 2y = x^2 - 2$ Sub. this in ② to get

$$x^2 - 2x = 0 \rightarrow x = 0 \text{ or } x = 2$$

When $x=0, y=-1 \rightarrow$ CP is $(0, -1)$ } $\therefore f$ has
 When $x=2, y=1 \rightarrow$ CP is $(2, 1)$ } two CPs.

$$\text{For the 2nd DT, } D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$$

$$\therefore D(x,y) = (2x)(2) - (-2)^2 = 4x - 4 \quad (3)$$

$D(0,-1) = -4 < 0$ so f has no relative extrema at $(0, -1)$. $\therefore f$ has a saddle point @ $(0, -1)$

$$D(2,1) = 8 - 4 = 4 > 0. \quad f_{xx}(2,1) = 8 > 0$$

$\therefore f$ has a relative minimum at $(2,1)$ of value $f(2,1)$.



#16 Examine the function $f(l,k) = l^2 + 4k^2 - 4lk$ for local extrema.

Solution : $f_l(l,k) = 2l - 4k = 0 \rightarrow l = 2k \leftarrow$ same
 $f_k(l,k) = 8k - 4l = 0 \rightarrow k = l \leftarrow$

\therefore the CP's of f occur at all points of the form $(2t, t)$ where $t \in \mathbb{R}$. (\because we have infinitely many CPs)
 Move to the 2nd DT:

$$D(l,k) = f_{ll}(l,k) f_{kk}(l,k) - (f_{lk}(l,k))^2 \\ = (2)(8) - (4)^2 = 0$$

$\therefore D(2t, t) = 0$ at every CP $(2t, t)$.

\therefore the 2nd DT is inconclusive in determining whether f has any extrema at CP's $(2t, t)$.

Remark: We can make a conclusion about extrema of f at pts $(2t, t)$ using "elementary algebra".

Factoring gives $f(l, k) = l^2 + 4k^2 - 4lk$ ④
 $= (l-2k)^2$

$\therefore f(2t, t) = 0$, so f is 0 at every CP $(2t, t)$

If $l \neq 2k$, then $f(l, k) > 0$. This means that f has relative min value of 0 at every CP $(2t, t)$. (In fact, f actually has absolute min value of 0 at every CP $(2t, t)$). 

#18 Examine the function $f(x, y) = (x-3)(y-3)(x+y-3)$ for relative extrema.

Solution : $f_x(x, y) = (y-3)(x+y-3) + (x-3)(y-3) = 0$
 $= (y-3)(2x+y-6) = 0$ ①

$$\begin{aligned} f_y(x, y) &= (x-3)(x+y-3) + (x-3)(y-3) = 0 \\ &= (x-3)(2y+x-6) = 0 \end{aligned}$$
②

We find all solutions to the system ① + ②.

In ①, if $y = 3$, then ② becomes

$$(x-3)(x) = 0 \text{ so } x = 3 \text{ or } x = 0.$$

\therefore this analysis shows $(0, 3)$ and $(3, 3)$ are CPs of f , so f has two CPs so far.

Now assume in ① that $y \neq 3$. We then have

$$2x+y-6=0 \text{ so } y=-2x+6. \text{ Using ②, if}$$

$x=3$, then $y=0$ so this yields the CP $(3, 0)$.

Lastly in ②, if $x \neq 3$, then $2y + x - 6 = 0$ ⑤
 $\text{so } (\because y = -2x + 6), -4x + 12 + x - 6 = 0$
 which gives $-3x + 6 = 0$ so $x = 2$ and $y = 2$.
 \therefore we have the final CP of $(2, 2)$.

Conclusion is that f has four CP's:
 $(0, 3), (3, 3), (3, 0), (2, 2)$.

Next we use the 2nd DT.

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 2(y-3)2(x-3) - [2x+y-6 + (y-3)]^2 \\ &= 4(x-3)(y-3) - [2x+2y-9]^2 \end{aligned}$$

(It is of no use to expand — leave factored!)

$D(0, 3) = 0 - 9 < 0 \rightarrow f$ has no extrema at $(0, 3)$, so $(0, 3)$ is a saddle point.

$D(3, 3) = 0 - 9 < 0 \rightarrow f$ has no extrema at $(3, 3)$, so $(3, 3)$ is another saddle point.

$D(3, 0) = 0 - 9 < 0 \rightarrow f$ has no extrema at $(3, 0)$, so $(3, 0)$ is yet a third saddle point.

$D(2, 2) = 4 - 1 = 3 > 0$, $f_{xx}(2, 2) = -2 < 0$,
 so f has a local maximum at $(2, 2)$ of value $f(2, 2)$.



#20 Examine the function $f(x,y) = \ln(xy) + 2x^2 - xy - 6x$ ⑥
for local extrema.

Solution: $f_x(x,y) = \frac{1}{x} + 4x - y - 6 = 0$ ①

$$f_y(x,y) = \frac{1}{y} - x = 0 \quad ②$$

From ② we have that $x = \frac{1}{y}$ and $\frac{1}{x} = y$

Sub these in ① to get $y + \frac{4}{y} - y - 6 = 0$

$\therefore y = \frac{2}{3}$ so $x = \frac{3}{2}$ so the only CP of f is $(\frac{3}{2}, \frac{2}{3})$.

$$D(x,y) = \left(-\frac{1}{x^2} + 4\right)\left(-\frac{1}{y^2}\right) - 1 \quad (\text{Again, there is no need to simplify})$$

$$D\left(\frac{3}{2}, \frac{2}{3}\right) = \left(\frac{32}{9}\right)\left(-\frac{9}{4}\right) - 1 = -9 < 0$$

$\therefore f$ has no extrema at $(\frac{3}{2}, \frac{2}{3})$ so that point is a saddle point of f . ■

#22 Solution is on page 14 14

#24 The profit per pound of A is $P_A - a$

" " " " " B is $P_B - b$

\therefore total profit (for selling A and B) is

$$P = (P_A - a)q_A + (P_B - b)q_B$$

$$= (P_A - a)[5(P_B - P_A)] + (P_B - b)[500 + 5(P_A - 2P_B)]$$

We now find the CP(s) of P .

$$\begin{aligned}\frac{\partial P}{\partial P_A} &= 5(P_B - P_A) - 5(P_A - a) + (P_B - b) 5 \quad (7) \\ &= -5(2P_A - 2P_B + b - a) = 0 \quad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial P_B} &= 5(P_A - a) + 500 + 5(P_A - 2P_B) + (-10)(P_B - b) \\ &= 5(2P_A - 4P_B + 2b - a + 100) = 0 \quad (2)\end{aligned}$$

From (1) we get $2P_A - 2P_B + b - a = 0$ and
from (2) we get $2P_A - 4P_B + 2b - a + 100 = 0$ (3)

Subtracting the bottom equation from the top one gives $2P_B - b - 100 = 0$ so $P_B = 50 + \frac{b}{2}$

Now substituting this into (3) gives

$$2P_A - 200 - 2b + 2b - a + 100 = 0 \text{ so}$$

$$2P_A - 100 - a = 0 \text{ hence } P_A = 50 + \frac{a}{2}$$

∴ the critical point for P is $(50 + \frac{a}{2}, 50 + \frac{b}{2})$

We use the 2nd DT to examine extrema.

$$D(P_A, P_B) = \frac{\partial^2 P}{\partial P_A^2} \frac{\partial^2 P}{\partial P_B^2} - \left(\frac{\partial^2 P}{\partial P_A \partial P_B} \right)^2$$

$$= (-10)(-20) - 10^2 = 100$$

$$\therefore D\left(50 + \frac{a}{2}, 50 + \frac{b}{2}\right) > 0 \quad \frac{\partial^2 P}{\partial P_A^2}\left(50 + \frac{a}{2}, 50 + \frac{b}{2}\right) < 0$$

$(= 100) \qquad \qquad \qquad (= -10)$

∴ by the 2nd DT, P has a relative maximum when $P_A = 50 + \frac{a}{2}$ and $P_B = 50 + \frac{b}{2}$ (8)

Maximum profit is $P\left(50 + \frac{a}{2}, 50 + \frac{b}{2}\right)$

#28 Solution is on page 15

#34 Assume (a, b) is a CP of a function $z = f(x, y)$ and all 2nd partial derivatives of f are continuous near (a, b) .

$$\text{Let } D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

and assume $D(a, b) > 0$.

(a) Show $f_{xx}(a, b) < 0$ iff $f_{yy}(a, b) < 0$

Proof: ∵ $D(a, b) > 0$, $f_{xx}(a, b)f_{yy}(a, b) - \underbrace{(f_{xy}(a, b))^2}_{\geq 0} > 0$

$$\therefore f_{xx}(a, b)f_{yy}(a, b) > 0$$

Thus, if $f_{xx}(a, b) < 0$ then $f_{yy}(a, b) < 0$ too

The converse is identical.

(b) Show $f_{xx}(a, b) > 0$ iff $f_{yy}(a, b) > 0$

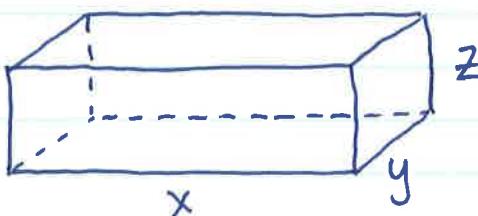
Proof: By the same calculation as in (a), we have that $f_{xx}(a, b)f_{yy}(a, b) > 0$

It is now clear that $f_{xx}(a, b) > 0$ iff $f_{yy}(a, b) > 0$ as required. ■

#36 Solution is on page 16

(9)

② Draw a typical rectangular box:



x = length

y = width

z = height

$x, y, z > 0$

For the girth we require that $x + 2y + 2z = 108$ (we can & should assume $= 108$ and not < 108 because, for < 108 , a dimension can always be increased to make $= 108$, and therefore increasing the volume). We maximize the volume $V = xyz$

Solution: 1st get V as a function of two variables only, $V(y, z)$, by letting $x = 108 - 2y - 2z$.

$$\begin{aligned}\therefore V(y, z) &= (108 - 2y - 2z)yz \\ &= 108yz - 2y^2z - 2yz^2\end{aligned}$$

Critical point(s):

$$V_y(y, z) = 108z - 4yz - 2z^2 = 0 \quad ①$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = 0 \quad ②$$

$$\because y, z > 0, \quad ① \rightarrow 108 - 4y - 2z = 0$$

$$\quad ② \rightarrow 108 - 2y - 4z = 0$$

1st eqⁿ $\rightarrow 2z = 108 - 4y$. Substitution into last eqⁿ gives $108 - 2y - 2(108 - 4y) = 0$

$\therefore -108 + 6y = 0 \rightarrow y = 18$ and ⑩
 $z = 18$ too. \therefore the only CP of V is $(18, 18)$

Check for maximum using 2nd DT:

$$D(y, z) = V_{yy}(y, z) V_{zz}(y, z) - (V_{yz}(y, z))^2$$

$$= (-4z)(-4y) - (108 - 4y - 4z)^2$$

$$\therefore D(18, 18) = (-72)^2 - (-36)^2 > 0$$

$$D V_{yy}(18, 18) = -72 < 0$$

$\therefore V$ is maximized when $y = z = 18$

$$\text{and } x = 108 - 36 - 36 = 36$$

$$\text{Maximum volume is } V(18, 18) = (36)(18)^2 \\ = 11,664 \text{ cm}^3$$



③ Given is the function $f(x, y) = x^2 - y^2 - 2x + 4y + 6$

(a) Analyzing the CP(s):

$$f_x(x, y) = 2x - 2 = 0 \rightarrow x = 1 \quad \therefore \text{the only CP of } f$$

$$f_y(x, y) = -2y + 4 = 0 \rightarrow y = 2 \quad \text{is } (1, 2)$$

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$$

$$= (2)(-2) = -4$$

$\therefore D(1, 2) = -4$, so the 2nd DT shows that f has no extrema at $(1, 2)$.

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(b) We show that f has no extrema at $(1, 2)$ purely "by algebraic methods".

Begin by factoring (and completing the square) the function f :

$$\begin{aligned} f(x,y) &= x^2 - 2x - (y^2 - 4y) + 6 \\ &= x^2 - 2x + 1 - (y^2 - 4y + 4) + 6 - 1 + 4 \\ &= (x-1)^2 - (y-2)^2 + 9 \end{aligned}$$

$f(1,2) = 9$. If $y \approx 2$, but $y \neq 2$, then

$f(1,y) = -(y-2)^2 + 9 < 9$. This means there are points of the form $(1,y)$ arbitrarily close to $(1,2)$ (but $\neq (1,2)$) for which $f(1,y) < 9$. $\therefore f$ cannot have a min at $(1,2)$. Similarly, if $x \approx 1$, but $x \neq 1$, $f(x,2) = (x-1)^2 + 9 > 9$, so f cannot have a max at $(1,2)$. We conclude that f has no extrema at $(1,2)$.



④ $f(x,y) = x^4 + y^4$

(a) $f_x(x,y) = 4x^3 = 0 \rightarrow x = 0 \quad \therefore (0,0)$ is the
 $f_y(x,y) = 4y^3 = 0 \rightarrow y = 0 \quad$ only CP of f .

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

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$$\therefore D(x,y) = (12x^2)(12y^2) \text{ so } D(0,0) = 0$$

\therefore 2nd DT is inconclusive - it tells nothing about extrema of f at $(0,0)$.

(b) We note that $f(0,0) = 0$ and that

$x^4, y^4 > 0$ so $f(x,y) = x^4 + y^4 > 0$. Since $f(0,0) = 0$ and $f(x,y) > 0$ if $(x,y) \neq (0,0)$, it follows that f has a local, and even a global, minimum at $(0,0)$ of value 0.

\therefore we analyzed "algebraically") 

⑤ Let $f(x,y) = x^4 - y^4$

$$(a) f_x(x,y) = 4x^3 = 0 \rightarrow x=0 \quad \therefore \text{the only CP of } f \text{ is } (0,0).$$

$$f_y(x,y) = 4y^3 = 0 \rightarrow y=0$$

$$D(x,y) = (12x^2)(12y^2) \text{ so } D(0,0) = 0$$

\therefore 2nd DT tells nothing about whether f has any extrema at $(0,0)$.

(b) $f(0,0) = 0$ If $x \approx 0$ but $x \neq 0$, then

$f(x,0) = x^4 > 0 \quad \therefore f$ cannot have a local maximum at $(0,0)$ If $y \approx 0$ but $y \neq 0$, then

$f(0,y) = -y^4 < 0 \quad \therefore f$ cannot have a min at $(0,0)$. 

$\therefore f$ has no extrema at $(0,0)$.

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⑥ Given is $f(x,y) = x^2 - e^{(y^2-1)}$

(a) $f_x(x,y) = 2x \quad f_y(x,y) = -e^{(y^2-1)}(2y)$

For CPs, we solve $f_x(x,y) = 0 + f_y(x,y) = 0$

We get the point $(0,0)$ as the only CP of f .

(b) $g(x) = f(x,x) = x^2 - e^{(x^2-1)}$ ($\because g$ is a function of one variable only)
 $g'(x) = 2x - e^{(x^2-1)}(2x)$
 $= 2x(1 - e^{(x^2-1)})$

$g'(x) = 0$ iff $x = 0, \pm 1$. Thus g has three CPs : $0, \pm 1$.

(c) It is odd that f (a function of two variables) has fewer critical points than g (which is a function of one variable only). It may have seemed intuitive that "more variables" \Rightarrow "more critical points". But (a) & (b) show this is false.

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Section 17.6 #22

Given is the function

$$Q = Q(c, d) = 18c + 20d - 2c^2 - 4d^2 - cd$$

c = # of hours computer C is used
 d = # of hours computer D is used

We maximize Q

Solution : $Q_c = \frac{\partial Q}{\partial c} = 18 - 4c - d = 0 \quad ①$

$$Q_d = \frac{\partial Q}{\partial d} = 20 - 8d - c = 0 \quad ②$$

$$① \Rightarrow d = 18 - 4c \text{ Sub in } ② \text{ to get}$$

$$20 - 8(18 - 4c) - c = 0$$

$$31c - 124 = 0 \rightarrow c = 4 \rightarrow d = 2$$

\therefore the only CP of Q is $(c, d) = (4, 2)$.

$$\begin{aligned} D(c, d) &= Q_{cc}(c, d) Q_{dd}(c, d) - [Q_{cd}(c, d)]^2 \\ &= (-4)(-8) - (-1)^2 = 31 > 0 \quad \forall (c, d) \end{aligned}$$

$$Q_{cc}(4, 2) = -4 < 0$$

$\therefore Q$ is maximized at the point $(4, 2)$ (this is a relative maximum). The relative maximum value is $Q(4, 2) = 56$. 

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Section 17.6 #28

We have products A and B.

Joint cost function is $C = 2(g_A + g_B + g_A g_B)$

Demand functions are $g_A = 20 - 2P_A$

$$g_B = 10 - P_B$$

P_A, P_B are unit prices of A and B

Profit = Revenue - Cost

$$P = P_A g_A + P_B g_B - C$$

$$= P_A (20 - 2P_A) + P_B (10 - P_B)$$

$$- 2[20 - 2P_A + 10 - P_B + (20 - 2P_A)(10 - P_B)]$$

Expansion, simplification gives

$$P = -2P_A^2 - P_B^2 - 4P_A P_B + 64P_A + 52P_B - 460$$

$$\frac{\partial P}{\partial P_A} = -4P_A - 4P_B + 64 = 0 \leftarrow \textcircled{1}$$

$$\frac{\partial P}{\partial P_B} = -4P_A - 2P_B + 52 = 0 \leftarrow \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \rightarrow -2P_B + 12 = 0 \rightarrow P_B = 6$$

$$\text{From } \textcircled{1} : -4P_A - 24 + 64 = 0 \rightarrow P_A = 10$$

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\therefore the only CP of P is $(10, 6)$

$$D(P_A, P_B) = \frac{\partial^2 P}{\partial P_A^2} \cdot \frac{\partial^2 P}{\partial P_B^2} - \left(\frac{\partial^2 P}{\partial P_A \partial P_B} \right)^2$$

$$= (-4)(-2) - (-4)^2 = -8 < 0$$

$\therefore P$ does not have any local extrema at the CP $(10, 6)$ (a surprise).

Using only the techniques developed in MATA33, we cannot say exactly what the maximum profit is for P .

Extra remarks: It turns out that P does have a maximum, but we do not find that using the ideas from MATA33. To get a bit of insight as to why P has a maximum, note first that P is a polynomial in 2 variables ($P_A + P_B$). The domain of P

is $\{(P_A, P_B) \mid 0 \leq P_A \leq 10 \text{ and } 0 \leq P_B \leq 10\}$
 $(P_A \leq 10 \text{ to guarantee } g_A > 0. \text{ Same for } P_B)$

Domain is a square and P is continuous. This will guarantee a maximum for P . 

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Section 17.6 #36

Profit from selling TV's is given as

$$P = 300 \left(\frac{7x}{2+x} + \frac{4y}{5+y} \right) - x - y$$

where x, y are weekly dollar expenditures on newspaper, radio advertising, respectively.

$$(\because x, y \geq 0)$$

We maximize P .

$$P_x = 300 \left(\frac{7(2+x) - 7x}{(2+x)^2} \right) - 1 = 0 \quad \leftarrow \textcircled{1}$$

$$P_y = 300 \left(\frac{4(5+y) - 4y}{(5+y)^2} \right) - 1 = 0 \quad \leftarrow \textcircled{2}$$

$$\textcircled{1} \rightarrow 4200 = (2+x)^2 \quad \textcircled{2} \rightarrow 6000 = (5+y)^2$$

$$\begin{aligned} \therefore x &= -2 \pm \sqrt{4200} \\ &= -2 \pm 10\sqrt{42} \end{aligned} \quad \begin{aligned} \therefore y &= -5 \pm \sqrt{6000} \\ &= -5 \pm 20\sqrt{15} \end{aligned}$$

$$\text{Only } x = -2 + 10\sqrt{42} \geq 0 \quad \text{Only } y = -5 + 20\sqrt{15} \geq 0$$

\therefore we have only one (relevant) CP is

$$(x, y) = (-2 + 10\sqrt{42}, -5 + 20\sqrt{15})$$

$$D(x, y) = P_{xx} \cdot P_{yy} - (P_{xy})^2 \quad P_{xy} = 0$$

$$= \left(-\frac{8400}{(2+x)^3} \right) \cdot \left(-\frac{12000}{(5+y)^3} \right)$$

(No need to simplify as we need only evaluate.)

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$$D(-2 + 10\sqrt{42}, -5 + 20\sqrt{15})$$

$$= \left(-\frac{8400}{(10\sqrt{42})^3} \right) \cdot \left(-\frac{12000}{(20\sqrt{15})^3} \right) > 0$$

$$P_{xx}(-2 + 10\sqrt{42}, -5 + 20\sqrt{15})$$

$$= -\frac{8400}{(10\sqrt{42})^3} < 0$$

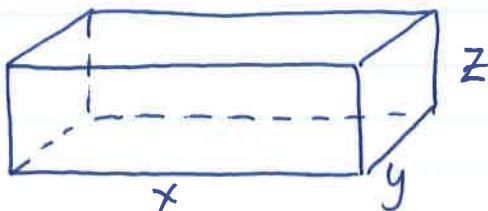
\therefore relative maximum profit occurs when

$$\begin{aligned} x &= -2 + 10\sqrt{42} \approx 62.81 \\ y &= -5 + 20\sqrt{15} \approx 72.46 \end{aligned} \quad \left. \begin{array}{l} \text{dollars per} \\ \text{week on} \\ \text{advertising} \end{array} \right.$$



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⑦



Length = $x > 0$
 Width = $y > 0$
 Height = $z > 0$

$$(a) \text{ Total Area} = 12 = xy + 2xz + 2yz$$

$$\therefore z = \frac{12 - xy}{2x + 2y}$$

$$\text{Volume } V = xyz = xy \left(\frac{12 - xy}{2x + 2y} \right)$$

$$(b) V = \frac{12xy - x^2y^2}{2x + 2y}$$

$$V_x = \frac{(2y - 2xy^2)(2x + 2y) - (12xy - x^2y^2)(2)}{(2x + 2y)^2} = 0$$

$$\therefore 24xy + 24y^2 - 4x^2y^2 - 4xy^3 - 24xy + 2x^2y^2 = 0$$

$$y^2(12 - 2xy - x^2) = 0 \quad \leftarrow ①$$

$$V_y = \frac{(12x - 2x^2y)(2x + 2y) - (12xy - x^2y^2)(2)}{(2x + 2y)^2} = 0$$

$$\therefore 24x^2 + 24xy - 4x^3y - 4x^2y^2 - 24xy + 2x^2y^2 = 0$$

$$\therefore x^2(12 - 2xy - y^2) = 0 \quad \leftarrow ②$$

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$\because x, y > 0$, ① + ② show that

$$12 - 2xy - x^2 = 0$$

$$12 - 2xy - y^2 = 0$$

$$\therefore x^2 = y^2 \quad \text{so} \quad x = y .$$

(c) When $x = y$, $12 - 3x^2 = 0$ from ① (or ②)

$$\therefore x^2 = 4 \quad \text{so} \quad x = 2 \quad \text{and} \quad y = 2$$

$$\text{Then } 12 = 4 + 4z + 4z \quad \text{so} \quad z = 1$$

So because of the "physical nature of the problem", max volume is assumed

$$\text{and it is } V = (2)(2)(1) = 4 \text{ m}^3$$

$$(c) V = \frac{12xy - x^2y^2}{2x+2y} \quad \text{From (b) we have } x = y$$

$$\therefore V = \frac{12x^2 - x^4}{4x} = 3x - \frac{1}{4}x^3$$

$$V'(x) = 3 - \frac{3}{4}x^2 = 0 \Rightarrow x = \pm 2$$

$\because x > 0$, we only consider $x = 2$

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By the 2nd DT (from MATA32),

$$V''(x) = -\frac{3}{2} < 0 \quad \text{so } V''(2) = -3 < 0$$

∴ V is (actually) maximized (and globally) when $x=2$.

Volume = 4 again.



(8)

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$f(x,y) = ax^2 + by^2 + cxy - x + y$ has a critical point at $(x,y) = (0,1)$ and the 2nd derivative test is inconclusive at $(0,1)$.

We find a, b, c .

Solution :

$$f_x(x,y) = 2ax + cy - 1$$

$$f_y(x,y) = 2by + cx + 1$$

$$\therefore 0 = f_x(0,1) = c - 1 \rightarrow \boxed{c = 1}$$

$$0 = f_y(0,1) = 2b + 1 \rightarrow \boxed{b = -\frac{1}{2}}$$

$$D(x,y) = (2a)(2b) - c^2 \rightarrow D(0,1) = 0 = -2a - 1 \rightarrow \boxed{a = -\frac{1}{2}}$$



Section 17.7 continued

(20)

28. The joint cost function is $c = 2(q_A + q_B + q_A q_B)$ with demand functions $q_A = 20 - 2P_A$ and $q_B = 10 - P_B$.

$$\text{Total Profit} = \text{Total Revenue} - \text{Total Cost} = P_A q_A + P_B q_B - c = P_A(20 - 2P_A) + P_B(10 - P_B) - 2(20 - 2P_A + 10 - P_B + (20 - 2P_A)(10 - P_B)) = -2P_A^2 - P_B^2 - 2P_A P_B + 42 P_A + 31 P_B + 230.$$

$$P_{q_A}(P_A, P_B) = -4P_A - 2P_B + 42 = 0 \Rightarrow P_B = 21 - 2P_A.$$

$$P_{q_B}(P_A, P_B) = -2P_A - 2P_B + 31 = 0$$

Substituting in the second equation $-2P_A - 2(21 - 2P_A) + 31 = 0 \Rightarrow 2P_A = 11 \Rightarrow P_A = 11/2$
 $\Rightarrow P_B = 21 - 2(11/2) = 10$. The critical point is $(11/2, 10)$.

Now $A = \begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$ so $\det A = 4 \neq 0$ and $\det A_1 = \det [-4] = -4 < 0$,
 $\det A_2 = \det [2] = 4 > 0$, so $(11/2, 10)$ gives a relative maximum.

The maximum profit occurs when $P_A = 5.5$, $P_B = 10$. With these prices we also have $q_A = 9$, $q_B = 0$ and profit = 40.5.

34. Suppose (a, b) is a critical point of a function $z = f(x, y)$ with continuous partial derivatives near (a, b) . Since $D(a, b) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}(a, b) f_{yy}(a, b)$

$-(f_{xy}(a, b))^2 \geq 0$ we have $f_{xx}(a, b) f_{yy}(a, b) \geq (f_{xy}(a, b))^2 \geq 0$

(a) Since $f_{xx}(a, b) f_{yy}(a, b) \geq 0$, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. Hence $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

(b) Since $f_{xx}(a, b) f_{yy}(a, b) \geq 0$, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. Hence $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

SOLUTIONS courtesy of Professor Moore.

(a) $f(x, y, z) = x^3 + xy^2 + x^2 + y^2 + 3z^2$

$$f_x(x, y, z) = 3x^2 + y^2 + 2x = 0$$

$$f_y(x, y, z) = 2xy + 2y = 0$$

$$f_z(x, y, z) = 6z = 0 \Rightarrow z = 0$$

Now the second $\Rightarrow y = 0$ or $x = -1$. If $y = 0$, the first becomes $3x^2 + 2x = x(3x + 2) = 0 \Rightarrow x = 0$ or $x = -2/3$. If $x = -1$, the first becomes $1 + y^2 = 0 \Rightarrow y^2 = -1$ which is impossible. \therefore The critical points are $(0, 0, 0)$ and $(-2/3, 0, 0)$.

$$\text{Now } A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 6x+2 & 2y & 0 \\ 2y & 2x+2 & 0 \\ 0 & 0 & 6 \end{bmatrix} = Hf(x, y, z)$$

At $(0, 0, 0)$: $\det A = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 24 \neq 0$. $\det A_1 = \det [2] = 2 > 0$, $\det A_2 = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$ and $\det A_3 = \det [2] = 2 > 0$. Hence $(0, 0, 0)$ is a relative minimum.

At $(-2/3, 0, 0)$: $\det A = \det \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = -8 \neq 0$. $\det A_1 = \det [-2] = -2 < 0$,

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(a) continued

$\det A_2 = \det \begin{bmatrix} -2 & 0 \\ 0 & z^3 \end{bmatrix} = -\frac{4}{3} < 0$ and $\det A_3 = \det A = -8 < 0$. Hence $(-\frac{2}{3}, 0, 0)$ is neither a relative maximum nor minimum.

$$(b) f(x,y,z) = x^3 + xz^2 - 3x^2 + y^2 + z^2$$

$$\begin{aligned} f_x(x,y,z) &= 3x^2 + z^2 - 6x = 0 \\ f_y(x,y,z) &= 2y = 0 \Rightarrow y = 0 \\ f_z(x,y,z) &= 2xz + 4z = 0 \end{aligned}$$

From the third, $2z(x+z) = 0 \Rightarrow z = 0$ or $x = -z$.

If $z = 0$, the first gives $0 = 3x^2 - 6x = 3x(x-2)$ so $x = 0$ or $x = 2$.

If $x = -z$, then $z^2 = -24$ which is impossible.

Hence the critical points are $(0,0,0)$ and $(2,0,0)$.

$$\text{Now } A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 6x-6 & 0 & 2x \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{bmatrix} = Hf(x,y,z)$$

$$\text{At } (0,0,0); \det A = \det \begin{bmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = -48 \neq 0. \det A_1 = \det [-6] = -6 < 0,$$

$\det A_2 = \det \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} = -12 < 0$ and $\det A_3 = \det A = -48 < 0$. Hence $(0,0,0)$ is neither a local maximum nor minimum.

$$\text{At } (2,0,0); \det A = \det \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 96 \neq 0. \det A_1 = \det [6] = 6 > 0,$$

$\det A_2 = \det \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = 12 > 0$ and $\det A_3 = \det A = 96 > 0$. Hence $(2,0,0)$ is a local minimum.

$$(c) f(x,y,z) = x^2y + y^2z + z^2 - 2x$$

$$f_x(x,y,z) = 2xy - 2 = 0$$

$$f_y(x,y,z) = x^2 + 2yz = 0$$

$$f_z(x,y,z) = y^2 + 2z^2 = 0 \Rightarrow z = -\frac{1}{2}y^2,$$

so the second $\Rightarrow y^2 = x^2$, so the first $\Rightarrow y = 1 \Rightarrow z = -\frac{1}{2}$. Since $xy = 1$ we have $x = 1$. The single critical point is $(1, 1, -\frac{1}{2})$.

$$\text{Now } A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{bmatrix} = Hf(x,y,z)$$

$$\text{At } (1, 1, -\frac{1}{2}); \det A = \det \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = -20 \neq 0. \det A_1 = \det [2] = 2 > 0,$$

$\det A_2 = \det \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} = -6 < 0$ and $\det A_3 = \det A = -20 < 0$. Hence $(1, 1, -\frac{1}{2})$ is neither a local maximum nor minimum.

$$(d) f(x,y,z) = xy - xz.$$

$$f_x(x,y,z) = y - z = 0 \text{ so we have } x = 0 \text{ and}$$

$$f_y(x,y,z) = x = 0$$

$$f_z(x,y,z) = -x = 0$$

$y = z$ for all $z \in \mathbb{R}$. The set of points $(0, z, z)$, $z \in \mathbb{R}$ are critical.

Now $A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ and $\det A = 0$. We have the degenerate case - the test gives us no conclusion.

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