

Solutions Assignment 8 MATA33

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① Section 17.4 Problems

#2 $f(x,y) = 2x^3y^2 + 6x^2y^3 - 3xy$

$$f_x(x,y) = 6x^2y^2 + 12x^2y^3 - 3y \quad f_{xx}(x,y) = 12xy^2 + 12y^3 \quad \blacksquare$$

#10 $z = \frac{\ln(x^2+5)}{y} \quad \frac{\partial z}{\partial x} = \frac{2x}{y(x^2+5)}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{2x}{y(x^2+5)} \right) = -\frac{2x}{y^2(x^2+5)} \quad \blacksquare$$

#16 $f(x,y) = x^3 - 6xy^2 + x^2 - y^3$

$$f_x(x,y) = 3x^2 - 6y^2 + 2x$$

$$f_{xy}(x,y) = -12y \quad \therefore f_{xy}(1,-1) = 12 \quad \blacksquare$$

#20 Let $f(x,y) = e^{xy}$

$$f_x(x,y) = ye^{xy} \quad f_{xx}(x,y) = y^2e^{xy}$$

$$f_{xy}(x,y) = e^{xy} + xye^{xy} = f_{yx}(x,y) \quad (\text{by The Equality of Mixed Partial Derivatives Theorem p. 763})$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y} (xe^{xy}) \\ = x^2e^{xy}$$

$$\therefore f_{xx} + f_{xy} + f_{yx} + f_{yy} = y^2e^{xy} + 2(1+xy)e^{xy} + x^2e^{xy} \\ = e^{xy}(x^2 + 2xy + y^2 + 2) \\ = f(x,y)((x+y)^2 + 2) \quad \text{as required.} \quad \blacksquare$$

#22 Assume the equation $z^3 - x^3 - x^2y - xy^2 - y^3 = 0$ (2) defines z implicitly as a function of x and y .

Write $\frac{\partial z}{\partial x}$ as z_x and $\frac{\partial^2 z}{\partial x^2}$ as z_{xx}

Differentiate to get $3z^2 z_x - 3x^2 - 2xy - y^2 = 0$

$$\therefore 3z^2 z_x = 3x^2 + 2xy + y^2$$

$$\therefore z_x = \frac{3x^2 + 2xy + y^2}{3z^2}$$

Differentiate again: $z_{xx} = \frac{(6x + 2y)(3z^2) - (3x^2 + 2xy + y^2)(6z z_x)}{(3z^2)^2}$
(w.r.t. x)

$$z_{xx} = \cancel{3z^2} \left[(6x + 2y)z - (3x^2 + 2xy + y^2)(2z_x) \right]$$

$$= \frac{6x^2 \cdot 3z^2}{(6x + 2y)z - (3x^2 + 2xy + y^2)(2 \frac{3x^2 + 2xy + y^2}{3z^2})}$$

$$= \frac{(6x + 2y)3z^3 - 2(3x^2 + 2xy + y^2)^2}{9z^5}$$



#24 Assume the equation $2z^2 = x^2 + 2xy + xz$ defines z implicitly as a function of x and y .

" $\frac{\partial}{\partial y}$ " yields $4zz_y = 2x + xz_y$ ($z_y = \frac{\partial z}{\partial y}$)

$$\therefore z_y = \frac{2x}{4z - x}$$

" $\frac{\partial}{\partial x}$ " yields $z_{yx} = \frac{2}{2y} \left(\frac{2x}{4z - x} \right)$

$$= \frac{2(4z - x) - 2x(4z_x - 1)}{(4z - x)^2}$$

$$= 2 \left[\frac{4z - x - 4xz_x + x}{(4z - x)^2} \right]$$

We next find z_x .

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$$4xz_x = 2x + 2y + z + xz_x$$

$$z_x = \frac{2x + 2y + z}{4z - x}$$

\therefore by (*),

$$\begin{aligned} z_{yx} &= 2 \left(\frac{4z-x - x \left[4 \left(\frac{2x+2y+z}{4z-x} \right) - 1 \right]}{(4z-x)^2} \right) \\ &= 2 \left(\frac{(4z-x)^2 - x[4(2x+2y+z) - (4z-x)]}{(4z-x)^3} \right) \\ &= 2 \left(\frac{16z^2 - 8zx + x^2 - 8x^2 - 8xy - x^2}{(4z-x)^3} \right) \\ &= 16 \left(\frac{2z^2 - zx - x^2 - xy}{(4z-x)^3} \right) \quad (\text{By the initial equation}) \\ &= 16 \left(\frac{x^2 + 2xy + xz - zx - x^2 - xy}{(4z-x)^3} \right) \\ &= \frac{16xy}{(4z-x)^3} \end{aligned}$$

(2) Section 17.5 Problems

#2 $z = 2x^2 + 3xy + 2y^2$, $x = r^2 - s^2$
 $y = r^2 + s^2$

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$$\begin{aligned}
 \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \\
 &= (4x+3y)(2r) + (3x+4y)(2r) \\
 &= 14r(x+y) \\
 &= 14r(r^2-s^2+r^2+s^2) = 28r^3
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\
 &= (4x+3y)(-2s) + (3x+4y)(2s) \\
 &= (-2x+2y)s = -2s(x-y) \\
 &= -2s(r^2-s^2-r^2-s^2) = 4s^3
 \end{aligned}$$



#4 $z = \sqrt{8x+y}$, $x = t^2 + 3t + 4$, $y = t^3 + 4$

$$\begin{aligned}
 \frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\
 &= \frac{8}{2\sqrt{8x+y}} (2t+3) + \frac{1}{2\sqrt{8x+y}} (3t^2) \\
 &= \frac{3t^2 + 16t + 24}{2\sqrt{8x+y}} \\
 &= \frac{3t^2 + 16t + 24}{2\sqrt{8(t^2+3t+4)+t^3+4}}
 \end{aligned}$$



(5)

#10 $w = \ln(xyz) \quad x = r^2s, \quad y = rs, \quad z = rs^2.$

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r} \\ &= \frac{yz}{xyz} \cdot (2rs) + \frac{xz}{xyz} \cdot (s) + \frac{xy}{xyz} (s^2) \\ &= \frac{2rs}{x} + \frac{s}{y} + \frac{s^2}{z} = \frac{2rs}{r^2s} + \frac{s}{rs} + \frac{s^2}{rs^2} = \frac{4}{r}\end{aligned}$$

#12 $y = 4 - x^2 \quad x = 2r + 3s - 4t \quad (\text{Note that } y \text{ is a function of 1-variable: } x)$

$$\frac{\partial y}{\partial t} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial t} = (-2x)(-4) = 8x = 8(2r + 3s - 4t)$$

#20 Given is $p = aP - whL$ where

$P = f(l, k)$, $l = Lg(h)$, and k is independent of L and h .

$$\begin{aligned}\frac{\partial p}{\partial L} &= a \frac{\partial P}{\partial L} - wh = a \left[\frac{\partial P}{\partial l} \cdot \frac{\partial l}{\partial L} + \frac{\partial P}{\partial k} \cdot \frac{\partial k}{\partial L} \right] - wh \\ &= a \left[\frac{\partial P}{\partial l} g(h) + \frac{\partial P}{\partial k} (0) \right] - wh \\ &= a \frac{\partial P}{\partial l} g(h) - wh\end{aligned}$$

(Remark: $\frac{\partial k}{\partial L} = 0$ above because k is independent of L).

$$\begin{aligned}
 \frac{\partial P}{\partial h} &= a \frac{\partial P}{\partial k} - \omega L = a \left[\frac{\partial P}{\partial l} \cdot \frac{\partial l}{\partial h} + \frac{\partial P}{\partial k} \cdot \frac{\partial k}{\partial h} \right] - \omega L \\
 &= a \left[\frac{\partial P}{\partial l} L g'(h) + \frac{\partial P}{\partial k}(0) \right] - \omega L \\
 &= a \frac{\partial P}{\partial l} L g'(h) - \omega L \quad \left(\frac{\partial k}{\partial h} = 0 \text{ as } k \text{ is independent of } h \right)
 \end{aligned} \tag{6}$$

(3) Problem from Page 795

#18 Assume the equation $z^2 + \ln(yz) + \ln(z) + x + z = 0$ defines z implicitly as a function of x and y . Find z_y .

It's easier to find z_y by simplifying first. We have

$$\begin{aligned}
 z^2 + \ln(y) + \ln(z) + \ln(z) + x + z &= 0 \\
 \therefore z^2 + \ln(y) + 2\ln(z) + x + z &= 0 \\
 \therefore 2zz_y + \frac{1}{y} + \frac{2z_y}{z} + z_y &= 0 \\
 z_y \left(2z + \frac{2}{z} + 1 \right) &= -\frac{1}{y} \\
 \therefore z_y &= -\frac{z}{y(2z^2 + 2 + z)} \quad \blacksquare
 \end{aligned}$$

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(4) We solve the equations $f_x(x,y) = 0 = f_y(x,y)$ for the functions below.

$$(a) f(x,y) = x^2 + 2y^2 - x^2y$$

$$f_x(x,y) = 2x - 2xy = 0 \quad (1)$$

$$f_y(x,y) = 4y - x^2 = 0 \quad (2)$$

$$(1) \rightarrow x(1-y) = 0 \text{ so } x=0 \text{ or } y=1$$

$$(2) \rightarrow y = \frac{x^2}{4}$$

\therefore solutions are $(0,0), (2,1), (-2,1)$.

$$(b) f(x,y) = xy + \frac{a^3}{x} + \frac{b^3}{y} \quad a,b \in \mathbb{R}, \neq 0$$

$$f_x(x,y) = y - \frac{a^3}{x^2} = 0 \quad (1)$$

$$f_y(x,y) = x - \frac{b^3}{y^2} = 0 \quad (2)$$

$$(1) \rightarrow y = \frac{a^3}{x^2} \text{ Substitution in (2) gives}$$

$$x - \frac{b^3 x^4}{a^6} = 0 \quad \therefore x\left(1 - \frac{b^3 x^3}{a^6}\right) = 0$$

$f(x,y)$ is undefined if $x=0$ or $y=0$, so we only consider $1 - \frac{b^3 x^3}{a^6} = 0$

$$\therefore x^3 = \frac{a^6}{b^3} \quad \text{so} \quad x = \frac{a^2}{b} \quad \text{and} \quad y = a^3 \left(\frac{b^2}{a^2}\right) = \frac{b^2}{a}$$

$$\therefore \text{the only solution is } \left(\frac{a^2}{b}, \frac{b^2}{a}\right)$$



(8)

$$\textcircled{5} \quad z = xe^y + ye^x \quad z_x = e^y + ye^x \\ z_y = xe^y + e^x$$

$$z_{xx} = ye^x \quad z_{xxx} = ye^x$$

$$z_{yy} = xe^y \quad z_{yyy} = xe^y$$

$$z_{xy} = e^y + e^x \quad z_{xyy} = e^y \quad z_{xxy} = e^x$$

$$\therefore z_{xxx} + z_{yyy} = ye^x + xe^y \\ xz_{xyy} + yz_{xxy} = xe^y + ye^x \quad \boxed{\text{equal}}$$

$$\textcircled{6} \quad \sum_{k=1}^n \frac{\partial^2 z}{\partial x_k^2} = \sum_{k=1}^n \frac{\partial}{\partial x_k} (e^u \cdot a_k) = \sum_{k=1}^n e^u \cdot (a_k)^2 \\ = e^u \left(\sum_{k=1}^n a_k^2 \right) = e^u = z \quad \boxed{\text{A}}$$

$$\textcircled{7} \quad z = x^2 + xy + y^2, \quad x = s+t, \quad y = st$$

(a) Obtain z_s, z_t by substitution first.

$$z(s,t) = (s+t)^2 + (s+t)st + (st)^2$$

$$\therefore z_s = 2(s+t) + st + (s+t)t + 2st^2$$

$$z_t = 2(s+t) + st + (s+t)s + 2s^2t$$

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(b) Obtain z_s , z_t using the chain rule.

$$\begin{aligned}
 \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\
 &= (2x + y)(1) + (x + 2y)(t) \\
 &= 2(s+t) + st + (s+t+2st)t \\
 &= 2(s+t) + st + (s+t)t + 2st^2 \quad (\text{as above in (a)})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \\
 &= (2x + y)(1) + (x + 2y)(s) \\
 &= 2(s+t) + st + (s+t+2st)s \\
 &= 2(s+t) + st + (s+t)s + 2s^2t \quad (\text{as above in (a)}) \quad \boxed{\text{Q}}
 \end{aligned}$$

$$(8) \quad z = \frac{x}{y} \quad x = se^t \quad y = 1 + se^{-t}$$

Repeat problem 7 for these functions.

$$(a) \quad z(s,t) = \frac{se^t}{1 + se^{-t}} = \frac{se^{2t}}{e^t + s} \quad (\text{Multiplied top/bottom by } e^t)$$

$$\therefore \frac{\partial z}{\partial s} = \frac{e^{2t}(e^t + s) - se^{2t}(1)}{(e^t + s)^2} = \frac{e^{3t}}{(e^t + s)^2}$$

$$\frac{\partial z}{\partial t} = \frac{2se^{2t}(e^t + s) - se^{2t}e^t}{(e^t + s)^2} = \frac{se^{3t} + 2s^2e^{2t}}{(e^t + s)^2}$$

(10)

$$\begin{aligned}
 (b) \quad \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\
 &= \left(\frac{1}{y}\right)(e^t) + \left(-\frac{x}{y^2}\right)(e^{-t}) \\
 &= \frac{ye^t - xe^{-t}}{y^2} = \frac{(1+se^{-t})e^t - se^t e^{-t}}{(1+se^{-t})^2} \\
 &= \frac{e^t + s - s}{(1+se^{-t})^2} = \frac{e^t}{e^{-2t}(e^t+s)^2} = \frac{e^{3t}}{(e^t+s)^2} \quad (\text{as above in (a)})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \\
 &= \left(\frac{1}{y}\right)(se^t) + \left(-\frac{x}{y^2}\right)(-se^{-t}) \\
 &= \frac{yse^t + xse^{-t}}{y^2} = \frac{(1+se^{-t})se^t + se^t se^{-t}}{(1+se^{-t})^2} \\
 &= \frac{(1+se^{-t})se^{3t} + s^2 e^{2t}}{(e^t+s)^2} \\
 &= \frac{se^{3t} + 2s^2 e^{2t}}{(e^t+s)^2} \quad (\text{as above in (a)}) \quad \blacksquare
 \end{aligned}$$

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