

Solutions ASSIGNMENT #7 MATA33

① Section 17.2 Problems

#2 Cost $c = 2x\sqrt{x+y} + 6,000$ We find $\frac{\partial c}{\partial x}(70, 74)$

$$\frac{\partial c}{\partial x} = 2\sqrt{x+y} + \frac{2x}{2\sqrt{x+y}} \quad (\text{No need to simplify})$$

$$\therefore \frac{\partial c}{\partial x}(70, 74) = 2\sqrt{144} + \frac{14.0}{2\sqrt{144}} = \frac{179}{6}$$

Thus, when production level is $x=70$ & $y=74$,
marginal cost with respect to x is $\frac{179}{6}$ ■

#4 Given production is $P = 15lk - 3l^2 + 5k^2 + 500$

Marginal productivity functions are

$$\frac{\partial P}{\partial l} = 15k - 6l \quad \text{and} \quad \frac{\partial P}{\partial k} = 15l + 10k$$



#6 The Cobb-Douglas production function is

$P = Al^\alpha k^\beta$ where $A, \alpha, \beta > 0$ are constants and $\alpha + \beta = 1$.

$$(a) \frac{\partial P}{\partial l} = A\alpha l^{\alpha-1} k^\beta = \frac{\alpha}{l} (Al^\alpha k^\beta) = \frac{\alpha P}{l}$$

$$(b) \frac{\partial P}{\partial k} = A\beta l^\alpha k^{\beta-1} = \frac{\beta}{k} (Al^\alpha k^\beta) = \frac{\beta P}{k}$$



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$$(C) l \frac{\partial P}{\partial l} + k \frac{\partial P}{\partial k} = l \left(\frac{\alpha P}{l} \right) + k \left(\frac{\beta P}{k} \right) \\ = (\alpha + \beta) P = 1 P = P$$



#8 Given demand functions are

$$q_A = 20 - P_A - 2P_B \quad q_B = 50 - 2P_A - 3P_B$$

$$\frac{\partial q_A}{\partial P_B} = -2 \quad \frac{\partial q_B}{\partial P_A} = -2 \quad \begin{array}{l} \text{Since these partials} \\ \text{are both } < 0, \\ A+B \\ \text{are complementary.} \\ (\text{See p. 758}) \end{array}$$

We also have $\frac{\partial q_A}{\partial P_A} = -1$ and $\frac{\partial q_B}{\partial P_B} = -3$

(as one would expect for demand functions).

#10 A production function for Canadian manufacturing is given as $P = 33l^{0.46}k^{0.52}$

$$\frac{\partial P}{\partial l} = 15.18l^{-0.54}k^{0.52} \quad \frac{\partial P}{\partial k} = 17.16l^{0.46}k^{-0.48}$$

$$\therefore \frac{\partial P}{\partial l}(1,1) = 15.18, \quad \frac{\partial P}{\partial k}(1,1) = 17.16$$



(3)

$$\#14 \text{ "General status" is } S_g = 7 S_e^{-\frac{1}{3}} S_i^{\frac{1}{2}}$$

where S_e represents status due to education and S_i represents status due to income.

$$\frac{\partial S_g}{\partial S_e} = 7 \left(\frac{1}{3}\right) S_e^{-\frac{2}{3}} S_i^{\frac{1}{2}} \quad \frac{\partial S_g}{\partial S_i} = \frac{7}{2} S_e^{\frac{1}{3}} S_i^{-\frac{1}{2}}$$

If $S_e = 125$ and $S_i = 100$, we have

$$\frac{\partial S_g}{\partial S_e} (125, 100) = \frac{7}{3} \frac{(100)^{\frac{1}{2}}}{(125)^{\frac{2}{3}}} = \left(\frac{7}{3}\right)\left(\frac{10}{25}\right) = \frac{14}{15} \approx 0.933$$

$$\frac{\partial S_g}{\partial S_i} (125, 100) = \frac{7}{2} \frac{(125)^{\frac{1}{3}}}{(100)^{\frac{1}{2}}} = \frac{35}{20} = \frac{7}{4} = 1.75$$

Interpretation: If S_e increases by 1 from 125 to 126, and S_i remains constant at 100, then S_g increases by about 0.933. If S_i increases by 1 from 100 to 101, and S_e remains constant at 125, then S_g increases by 1.75. 

#18 Demand functions are

$$q_A = e^{-(P_A + P_B)} \quad q_B = 16 P_A^{-2} P_B^{-2}$$

A, B are products with unit prices P_A, P_B resp.

(4)

$$(a) \frac{\partial q_A}{\partial P_B} = -e^{-(P_A + P_B)} < 0 \quad \frac{\partial q_B}{\partial P_A} = -32 P_A^{-3} P_B^{-2} < 0$$

\therefore both derivatives above are < 0 , $A + B$ are complementary. (p. 758)

(b) Assume unit prices are 1 and 2 (\$1,000's) respectively for A and B (Note that unit prices are given in \$1,000's)

$\therefore P_A = 1, P_B = 2$. We then have that

$$\frac{\partial q_A}{\partial P_B} = -e^{-3} \text{ A decrease in the price of B of } \$20$$

represents a decrease in the quantity P_B by $\frac{20}{2000} = 0.01 \therefore$ the change in q_A is about

$$(-e^{-3})(-0.01) = \frac{0.01}{e^3}, \text{ so demand for A increases about } \frac{0.01}{e^3} \approx 0.0005 \text{ unit.}$$



#20 Given joint cost function is

$$C = \frac{q_A^2 (q_B^3 + q_A)}{17} + q_A q_B^{\frac{1}{3}} + 600$$

where q_A = amount of A produced

q_B = " " B "

(a) We find $\frac{\partial C}{\partial q_A}$ and $\frac{\partial C}{\partial q_B}$.

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$$(a) \frac{\partial c}{\partial q_A} = \frac{1}{17} \left\{ 2q_A (q_B^3 + q_A)^{-1/2} + q_A^2 \frac{1}{2} (q_B^3 + q_A)^{-3/2} \right\} + q_B^{1/3}$$

$$\frac{\partial c}{\partial q_B} = \frac{1}{17} \frac{1}{2} q_A^2 (q_B^3 + q_A)^{-1/2} (3q_B^2) + \frac{1}{3} q_A q_B^{-2/3}$$

(b) When $q_A = 17$ and $q_B = 8$, we get

$$\frac{\partial c}{\partial q_A} (17, 8) = \frac{1}{17} \left\{ 34 (8^3 + 17)^{-1/2} + \frac{(17)^2}{2} (8^3 + 17)^{-3/2} \right\} + 2 \approx 48.37$$

(c) If q_A is reduced from 17 to 16 (a reduction of 1 unit) and q_B is fixed @ 8 units, then the cost decreases by about \$48.37. 

② Section 17.3 Problems

#2 Given equation is $z^2 - 5x^2 + y^2 = 0$

We assume z is defined implicitly as a function of x and y . Write $z_x = \frac{\partial z}{\partial x}$ (easy notation)

$$\therefore \frac{\partial}{\partial x} (z^2 - 5x^2 + y^2) = 0$$

$$2zz_x - 10x = 0 \quad \therefore z_x = \frac{10x}{2z} = \frac{5x}{z}$$



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#6 Given equation is $z^3 + 2x^2z^2 - xy = 0$

Write z_x as $\frac{\partial z}{\partial x}$

We get by differentiation (partial) w.r.t. x :

$$3z^2 z_x + 4x z^2 + 4x^2 z z_x - y = 0$$

$$\therefore z_x (3z^2 + 4x^2 z) = y - 4x z^2$$

$$\therefore z_x = \frac{y - 4x z^2}{3z^2 + 4x^2 z}$$



#10 Given equation is $\ln(x) + \ln(y) - \ln(z) = e^y$
 We find $\frac{\partial z}{\partial x}$. Let that be z_x ($\stackrel{i.e.}{=} \text{again let } \frac{\partial z}{\partial x} = z_x$)

Differentiation w.r.t. x gives:

$$\frac{1}{x} - \frac{z_x}{z} = 0 \rightarrow z_x = \frac{z}{x}$$



#14 Given equation is $e^{zx} = xyz$. We find

$$\frac{\partial z}{\partial y} \Big|_{\substack{x=1 \\ y=-\frac{1}{e} \\ z=-1}} \quad (\stackrel{i.e.}{=} z_y(1, -\frac{1}{e}, -1))$$

Differentiation of both sides of the equation above w.r.t. y gives

$$\frac{\partial}{\partial y}(e^{zx}) = \frac{\partial}{\partial y}(xyz)$$

$$e^{zx}(z_y x) = xz + xyz_y$$

(Remember:

z is assumed to be a function of x & y
 And neither x nor y is a function of y or x , respectively)

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$$\therefore z_y(xe^{zx} - xy) = xz$$

$$\therefore z_y = \frac{xz}{xe^{zx} - xy} = \frac{z}{e^{zx} - y}$$

$$z_y(1, -\frac{1}{e}, -1) = \frac{-1}{e^{-1} + e^{-1}} = -\frac{e}{2}$$



#1b Given equation is $(xz+y^2)^{1/2} - xy = 0$.

$$\text{We find } z_y(2, 2, 6)$$

Differentiate w.r.t. y to get

$$\frac{1}{2}(xz+y^2)^{-1/2}(x z_y + 2y) - x = 0$$

$$\therefore z_y \frac{x}{2\sqrt{xz+y^2}} = x - \frac{y}{\sqrt{xz+y^2}}$$

$$\therefore z_y(2, 2, 6) = \frac{\frac{2}{2} - \frac{2}{4}}{\frac{2}{8}} = 4\left(\frac{3}{2}\right) = 6$$



③ Find the equation of the horizontal plane that is tangent to the graph of the function

$$G(x, y) = x^2 - 4xy - 2y^2 + 12x - 12y - 1$$

Solution: A plane is horizontal if and only if it has the form $z = k$ for some real constant k . $\therefore z$ is independent of x & y , so (p. 119)

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$$G_x(x,y) = 0 \quad \text{and} \quad G_y(x,y) = 0.$$

$$G_x(x,y) = 2x - 4y + 12 = 0 \quad ①$$

$$G_y(x,y) = -4x - 4y - 12 = 0 \quad ②$$

$$① - ② \text{ gives } 6x + 24 = 0 \rightarrow x = -4$$

Sub-back into ① to get $-8 - 4y + 12 = 0$ so $y = 1$
We obtain k by substitution :

$$G(-4, 1) = 16 + 16 - 2 - 48 - 12 - 1 = -31$$

∴ equation of the plane is $z = -31$

and it is tangent to the graph of G at the point $(-4, 1, -31)$. 

$$④ P(x,y) = \frac{xy}{ax+by} \quad \text{where } x,y > 0 \text{ are product}$$

X, Y sales (resp.), and $a, b > 0$ are constants.

$$P_x(x,y) = \frac{y(ax+by) - xy a}{(ax+by)^2} = \frac{by^2}{(ax+by)^2}$$

$$P_y(x,y) = \frac{x(ax+by) - xy b}{(ax+by)^2} = \frac{ax^2}{(ax+by)^2}$$

When $x=y$, the sum is

$$P_x(x,x) + P_y(x,x) = \frac{bx^2 + ax^2}{(ax+bx)^2} = \frac{x^2(a+b)}{x^2(a+b)^2} = \frac{1}{a+b}$$


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⑤ Given cost function is $C = 3x + 0.05xy + 9y + 500$

$$(a) C_x(x, y) = 3 + 0.05y \quad C_x(50, 100) = 8$$

$$(b) C(51, 100) - C(50, 100) \\ = 153 + 255 + 900 + 500 \\ - 150 - 250 - 900 - 500 = 8$$

$$\therefore C_x(50, 100) \approx C(51, 100) - C(50, 100)$$

↑ Actually, we get exact

$$C_y(x, y) = 0.05x + 9 \quad C_y(50, 100) = 11.5$$

$$C(50, 101) - C(50, 100) \\ = 150 + 252.5 + 909 + 500 \\ - 150 - 250 - 900 - 500 = 11.5$$

$$\therefore C_y(50, 100) \approx C(50, 101) - C(50, 100)$$

↑ Again, we get exact.

$$(c) C_x(a, b) = 3 + .05b$$

$$C(a+1, b) - C(a, b)$$

$$= 3(a+1) + .05(a+1)b + 9b + 500 \\ - 3a - .05ab - 9b - 500$$

$$= 3 + .05b$$

$$\therefore C_x(a, b) = C(a+1, b) - C(a, b)$$

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$$C_y(a, b) = 0.05a + 9$$

$$C(a, b+1) - C(a, b)$$

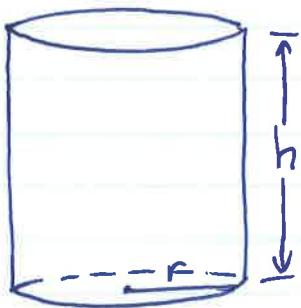
$$= 3a + 0.05a(b+1) + 9(b+1) + 500 \\ - 3a - 0.05ab - 9b - 500$$

$$= 0.05a + 9$$

$$\therefore C_y(a, b) = C(a, b+1) - C(a, b)$$



⑥ (a)



$r > 0$ is the radius of the cylinder

$h > 0$ is the height of the cylinder

$$K(h, r, a, b, \omega) = (\text{top cost}) + (\text{bottom cost}) + (\text{wall cost}) \\ = \pi r^2 a + \pi r^2 b + 2\pi r h \omega$$

$$(b) K_h = 2\pi r \omega, K_r = 2\pi r a + 2\pi r b + 2\pi h \omega$$

$$K_a = \pi r^2, K_b = \pi r^2, K_\omega = 2\pi r h$$

$$(c) K(h, r, 1, 2, 1) = \pi r^2 + 2\pi r^2 + 2\pi r h \\ = 3\pi r^2 + 2\pi r h$$

$$(d) L(\pi) = \left\{ (h, r) \mid K(h, r, 1, 2, 1) = \pi \right\}$$

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$$\therefore 3\pi r^2 + 2\pi r h = \pi$$

We get the equation $3r^2 + 2rh = 1$

Solve for h : $2rh = 1 - 3r^2$

$$\therefore h = \frac{1 - 3r^2}{2r}$$

We have that both $r > 0$ and $h > 0$

\therefore we require that $1 - 3r^2 > 0$

so $r < \frac{1}{\sqrt{3}}$. The desired inequality
is thus $0 < r < \frac{1}{\sqrt{3}}$



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