Solutions Assignment #6 MATA33

① Section 2.8 Problems

#2 $f(x,y) = 3x^2y - 4y \quad f(2,-1) = 3(2)^2(-1) - 4(-1) = -8$ ■

#4 $g(x,y,z) = x^2yz + xy^2z + xyz^2$

$g(3,1,-2) = (3)^2(1)(-2) + (3)(1)^2(-2) + (3)(1)(-2)^2$

$= -18 - 6 + 12 = -12$ ■

#8 $g(P_A, P_B) = P_A^2\sqrt{P_B} + 9 \quad g(4,9) = 4^2\sqrt{9} + 9 = 57$ ■

#10 $F(x,y,z) = \dfrac{2x}{(y+1)z} \quad F(1,0,3) = \dfrac{2}{(1)(3)} = \dfrac{2}{3}$ ■

#12 $f(x,y) = x^2y - 3y^3 \quad f(r+t, r) = (r+t)^2 r - 3r^3$

$= r^3 + 2r^2t + t^2r - 3r^3$

$= r(t^2 + 2rt - 2r^2)$ ■

#16 A plane parallel to the y,z-plane has the form $x = k$ where k is a constant (see page 119 and the diagrams there).

$\because$ $(-2,0,0)$ is on the plane, $k=2$ so the equation is $x = -2$. ■

#18 Any plane parallel to the y,z-plane has (as in #16) general equation $x = k$, k is a constant. $\because$ the point $(96,-2,2)$ is on the plane, we have $k = 96$, so the equation is $x = 96$. ■

#20 We sketch the surface $2x+y+2z=6$.

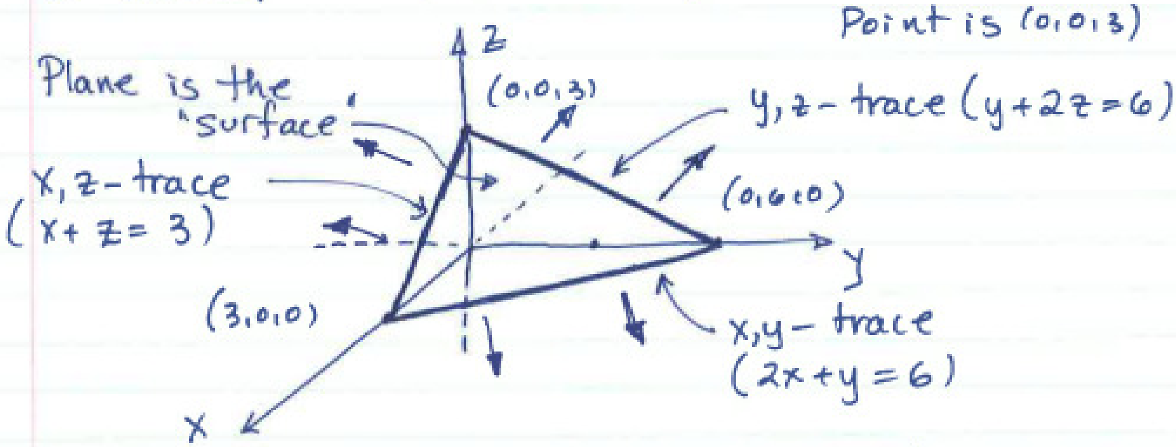
Solution: The surface is a plane because its equation has the form of the equation of a plane: $Ax+By+Cz+D=0$ (p.119)

The plane intersects the coordinate axis at the intercepts.

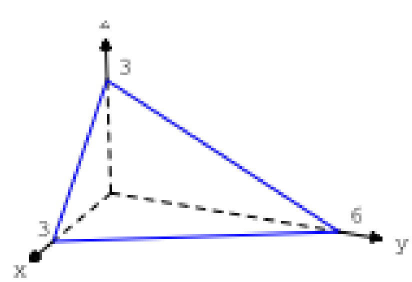
x-intercept → sub-in $y=z=0$ → $x=3$
Point is $(3,0,0)$

y-intercept → sub-in $x=z=0$ → $y=6$
Point is $(0,6,0)$

z-intercept → sub-in $x=y=0$ → $z=3$
Point is $(0,0,3)$



Only that part of the plane in the 1[st] octant is shown. The plane continues in the directions of the arrows.



#24 We sketch the surface $y = 3z + 2$.

Solution: The surface is a plane because it can be written in the form of a plane in $\mathbb{R}^3$: $Ax + By + Cz + D = 0$ (p.119)
Here: $A = 0$, $B = 1$, $C = -3$, $D = -2$

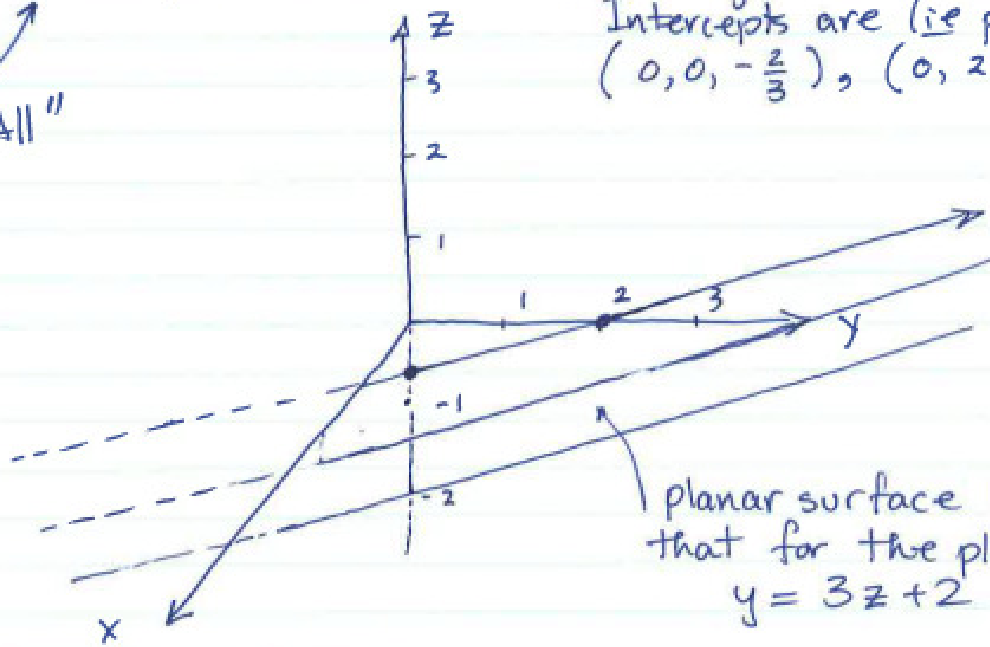
(3)

The y,z-trace is $y = 3z + 2$ (that is the line of intersection of the plane and the y,z-plane). Since x is missing from the given equation, x can assume any real value.

$\therefore\ \forall x \in \mathbb{R}$, we have the line $y = 3z + 2$

"For All"

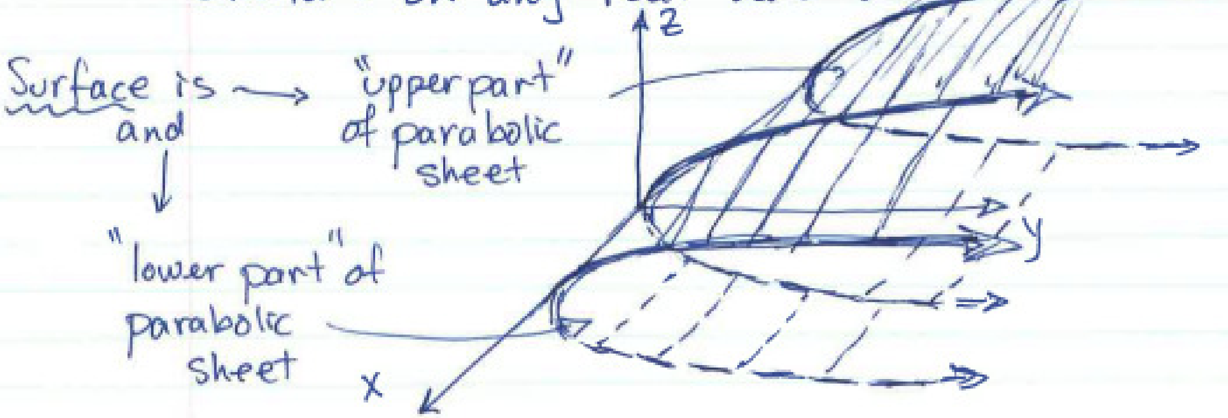
Intercepts are (ie points) $(0, 0, -\frac{2}{3})$, $(0, 2, 0)$

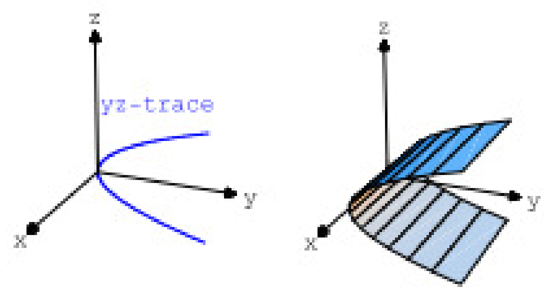


planar surface is that for the plane $y = 3z + 2$

#26 Sketch the surface $y = z^2$

Solution: The y,z-trace is the parabola $y = z^2$. Since x does not appear in the equation, x can take on any real value.

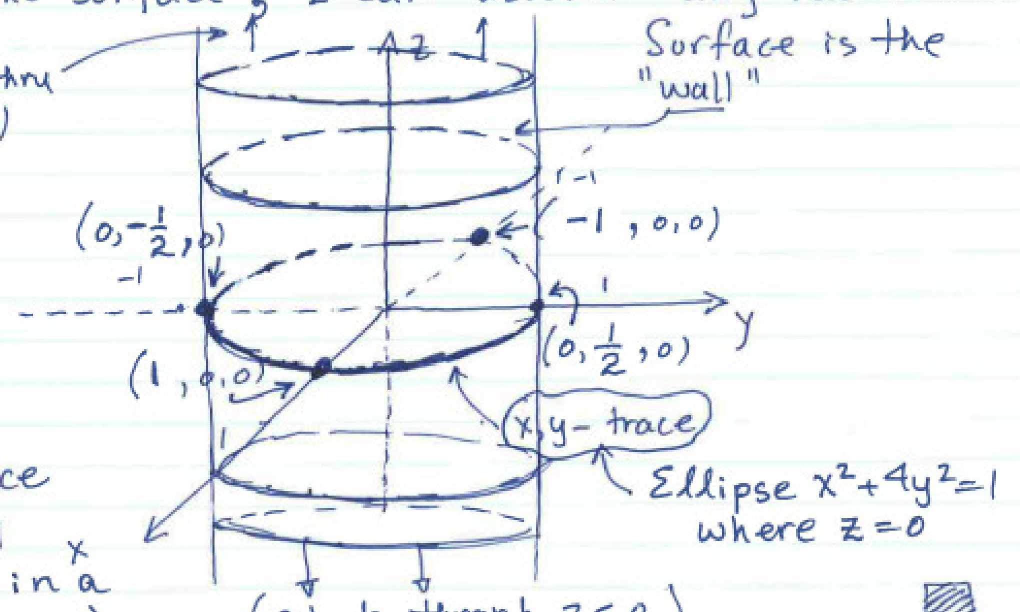




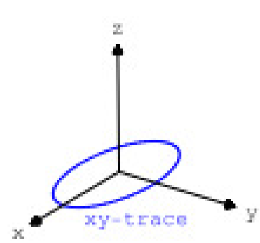
#28 We sketch the surface $x^2+4y^2=1$

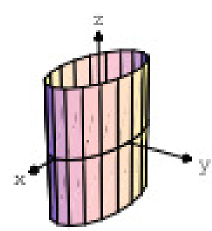
Solution: The "x, y-trace" is the ellipse $x^2+4y^2=1$ (this is the intersection of our surface $x^2+4y^2=1$ and the x, y-plane (i.e. $z=0$))

Since z does not appear in the equation of the surface, z can assume any real value.



(The surface $3x^2+2y^2=1$ is drawn in a similar way)





② Section 17.1 Problems

#2 $f(x,y)=2x^2+3xy$ $\quad f_x(x,y)=4x+3y$

$f_y(x,y)=3x$

#6 $g(x,y)=(x^2+1)^2+(y^3-3)^2+5xy^3-2x^2y^2$

$g_x(x,y)=2(x^2+1)(2x)+5y^3-4xy^2$ $\quad g_y(x,y)=2(y^3-3)(3y^2)+15xy^2-4x^2y$

Can also write g_x as $\frac{\partial g}{\partial x}(x,y)$ and g_y as $\frac{\partial g}{\partial y}(x,y)$

#8 $g(w,z) = \sqrt[3]{w^2+z^2} = (w^2+z^2)^{1/3}$

$$g_w(w,z) = \frac{1}{3}(w^2+z^2)^{-\frac{2}{3}}(2w) = \frac{2w}{3(w^2+z^2)^{2/3}}$$

$$g_z(w,z) = \frac{1}{3}(w^2+z^2)^{-\frac{2}{3}}(2z) = \frac{2z}{3(w^2+z^2)^{2/3}}$$

Can also write $\frac{\partial g}{\partial w}(w,z)$ for $g_w(w,z)$ and $\frac{\partial g}{\partial z}(w,z)$ for $g_z(w,z)$. ▨

#10 $h(u,v) = \dfrac{8uv^2}{u^2+v^2}$

$$h_u(u,v) = \frac{8v^2(u^2+v^2) - 8uv^2(2u)}{(u^2+v^2)^2} = \frac{8v^2(u^2+v^2-2u^2)}{(u^2+v^2)^2}$$

$$= \frac{8v^2(v^2-u^2)}{(u^2+v^2)^2}$$

$$h_v(u,v) = \frac{16uv(u^2+v^2) - 8uv^2(2v)}{(u^2+v^2)^2} = \frac{8uv(2u^2+2v^2-2v^2)}{(u^2+v^2)^2}$$

$$= \frac{16u^3v}{(u^2+v^2)^2}$$ ▨

#14 $h(x,y) = \dfrac{\sqrt{x+9}}{x^2y+y^2x} = \dfrac{(x+9)^{1/2}}{x^2y+y^2x}$

$$h_x(x,y) = \frac{\frac{1}{2}(x+9)^{-1/2}(x^2y+y^2x) - (x+9)^{1/2}(2xy+y^2)}{(x^2y+y^2x)^2}$$

$$h_y(x,y) = (x+9)^{1/2}(-1)(x^2y+y^2x)^{-2}(x^2+2xy)$$

$$= -\frac{(x+9)^{1/2}(x^2+2xy)}{(x^2y+y^2x)^2} = \frac{-\sqrt{x+9}\,(x+2y)}{xy^2(x+y)^2}$$

$\blacksquare$

#16 $z = (x^3+y^3)\,e^{xy+3x+3y}$

$$z_x = \frac{\partial z}{\partial x} = 3x^2 e^{xy+3x+3y} + (x^3+y^3)e^{xy+3x+3y}(y+3)$$

$$= \left(3x^2 + (x^3+y^3)(y+3)\right)e^{xy+3x+3y}$$

$$z_y = \frac{\partial z}{\partial y} = 3y^2 e^{xy+3x+3y} + (x^3+y^3)e^{xy+3x+3y}(x+3)$$

$$= \left(3y^2 + (x^3+y^3)(x+3)\right)e^{xy+3x+3y}$$

$\blacksquare$

#20 $f(r,s) = \sqrt{rs}\,e^{2+r} = r^{1/2}s^{1/2}e^{2+r}$

$$f_r(r,s) = \frac{1}{2}r^{-1/2}s^{1/2}e^{2+r} + r^{1/2}s^{1/2}e^{2+r}$$

$$= \left(\sqrt{rs} + \frac{1}{2}\sqrt{\frac{s}{r}}\right)e^{2+r}$$

$$f_s(r,s) = \frac{1}{2}r^{1/2}s^{-1/2}e^{2+r} = \frac{1}{2}\sqrt{\frac{r}{s}}\,e^{2+r}$$

$\blacksquare$

#28 $z=\sqrt{2x^3+5xy+2y^2} = (2x^3+5xy+2y^2)^{1/2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(2x^3+5xy+2y^2)^{-1/2} \cdot (6x^2+5y)$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}} = \frac{5}{2\sqrt{2}}$$

(∵ we are evaluating there is no need to simplify)

#30 $g(x,y,z) = \dfrac{3x^2y^2+2xy+x-y}{xy-yz+xz}$

(Similar comment as above in #28 regarding evaluation & no simplify)

$$g_y(x,y,z) = \frac{(6x^2y+2x-1)(xy-yz+xz) - (3x^2y^2+2xy+x-y)(x-z)}{(xy-yz+xz)^2}$$

$$\therefore g_y(1,1,5) = \frac{(6+2-1)(1-5+5) - (3+2+1-1)(1-5)}{(1-5+5)^2}$$

$$= 27$$

#36 $u = f(t,r,z) = \dfrac{(1+r)^{1-z}\ln(1+r)}{(1+r)^{1-z}-t}$

We calculate $\dfrac{\partial u}{\partial z} = u_z$ and verify the claim

$$u_z = \frac{t(1+r)^{1-z}\ln^2(1+r)}{[(1+r)^{1-z}-t]^2}.$$

⑧

Re-write u as $u = \ln(1+r)\left[\frac{(1+r)^{1-z}}{(1+r)^{1-z} - t}\right]$

$$\therefore u_z = \ln(1+r)\left[\frac{(1+r)^{1-z}\ln(1+r)(-1)\left[(1+r)^{1-z} - t\right] + (1+r)^{1-z}(1+r)^{1-z}\ln(1+r)}{\left[(1+r)^{1-z} - t\right]^2}\right]$$

$$= \frac{\ln^2(1+r)(1+r)^{1-z}\left[-(1+r)^{1-z} + t + (1+r)^{1-z}\right]}{\left[(1+r)^{1-z} - t\right]^2}$$

$$= \frac{t\,(1+r)^{1-z}\ln^2(1+r)}{\left[(1+r)^{1-z} - t\right]^2}$$ as required. ▨

#38 $r_L = r + D\frac{\partial r}{\partial D} + \frac{dC}{dD}$ is given

Elasticity is $\eta = \frac{r/D}{\partial r/\partial D}$. We verify that
(Given)

$$r_L = r\left[\frac{1+\eta}{\eta}\right] + \frac{dC}{dD}$$

Solution: $\eta = \frac{r/D}{\partial r/\partial D} \rightarrow \frac{\partial r}{\partial D} = \frac{r}{D\eta}$

Substitute to get $r_L = r + D\frac{r}{D\eta} + \frac{dC}{dD}$

$$= r\left[1 + \frac{r}{\eta}\right] + \frac{dC}{dD} = r\left[\frac{1+\eta}{\eta}\right] + \frac{dC}{dD}.$$ ▨

#34 Given is $z = \dfrac{x^2+y^2}{e^{x^2+y^2}}$

$$\frac{\partial z}{\partial x} = \frac{2x\left(e^{x^2+y^2}\right) - (x^2+y^2)e^{x^2+y^2}(2x)}{\left(e^{x^2+y^2}\right)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y\left(e^{x^2+y^2}\right) - (x^2+y^2)e^{x^2+y^2}(2y)}{\left(e^{x^2+y^2}\right)^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}} = \frac{(0)(e^0) - (0)e^0(0)}{(e^0)^2} = 0$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = \frac{2e^2 - 2e^2(2)}{(e^2)^2} = -\frac{2}{e^2}$$

$\blacksquare$

#35 (As an extra solution)

$$z = xe^{x-y} + ye^{y-x}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + xe^{x-y} + ye^{y-x}(-1) + xe^{x-y}(-1) + e^{y-x} + ye^{y-x}$$

$$= e^{x-y} + e^{y-x}$$ as required.

$\blacksquare$

(3) Amount of an Annuity formula is

$$S = f(R, r, n) = R\left[\frac{(1+r)^n - 1}{r}\right]$$

$$\frac{\partial S}{\partial R} = \frac{(1+r)^n - 1}{r}$$

$$\frac{\partial S}{\partial r} = R\left[\frac{n(1+r)^{n-1}r - ((1+r)^n - 1)}{r^2}\right]$$

$$= \frac{R}{r^2}\left[n(1+r)^{n-1}r - (1+r)^n + 1\right]$$

$$\frac{\partial S}{\partial n} = \frac{R}{r}(1+r)^n \ln(1+r)$$

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(4) $z = f(x,y) = x^2 - y + 1$

(a) Level curves are determined by the equation $f(x,y) = c$, $c \in \mathbb{R}$ is a constant.

For $L(0)$, $c = 0$, so we get $x^2 - y + 1 = 0$

hence $y = x^2 + 1$ gives the points $(x,y) \in L(0)$.

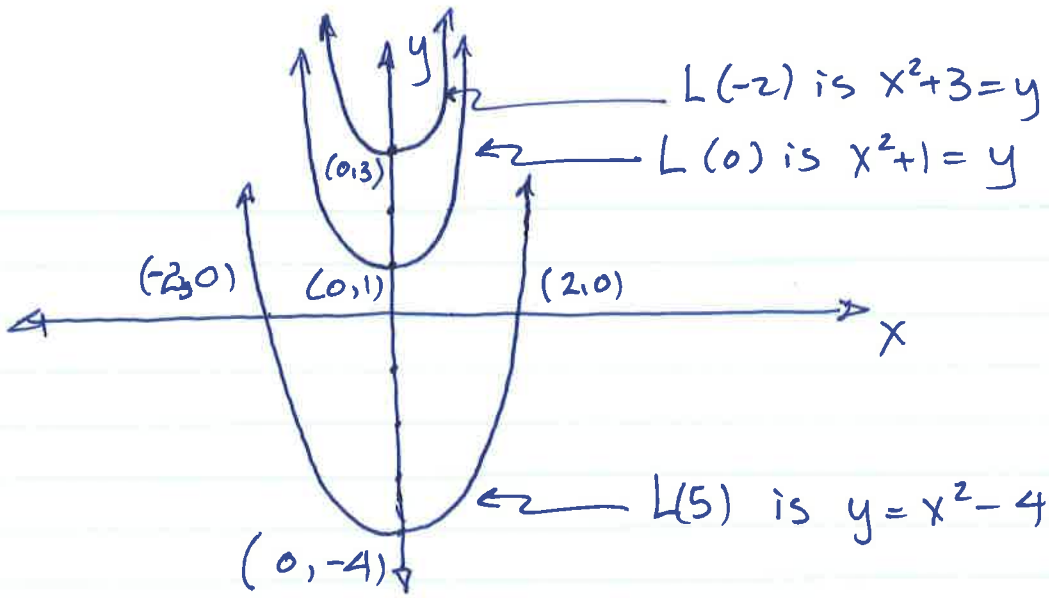
For $L(-2)$, $c = -2$, so we get $x^2 - y + 1 = -2$

hence $y = x^2 + 3$ gives the points $(x,y) \in L(-2)$.

For $L(5)$, $c = 5$, so we get $x^2 - y + 1 = 5$

hence $y = x^2 - 4$ gives the points $(x,y) \in L(5)$.

These are graphed together on the next page.



(b) $g(x,y) = f(f(x,y),y) = f(x^2-y+1, y)$

$= (x^2-y+1)^2 - y + 1$

$= (x^2-y+1)(x^2-y+1) - y + 1$

$= x^4 - x^2y + x^2 - yx^2 + y^2 - y + x^2 - y + 1 - y + 1$

$= x^4 - 2x^2y + 2x^2 - 3y + y^2 + 2$

$h(x,y) = f(x, f(x,y)) = f(x, x^2-y+1)$

$= x^2 - (x^2-y+1) + 1 = x^2 - x^2 + y - 1 + 1 = y$

(c) Assume $x \neq y$ and $f(x,y) = f(y,x)$

$\therefore\ x^2 - y + 1 = y^2 - x + 1$

$\therefore\ x^2 - y^2 + x - y = 0 \rightarrow (x-y)(x+y+1) = 0$

$\because\ x \neq y$, $x - y \neq 0$, so $x+y+1=0$

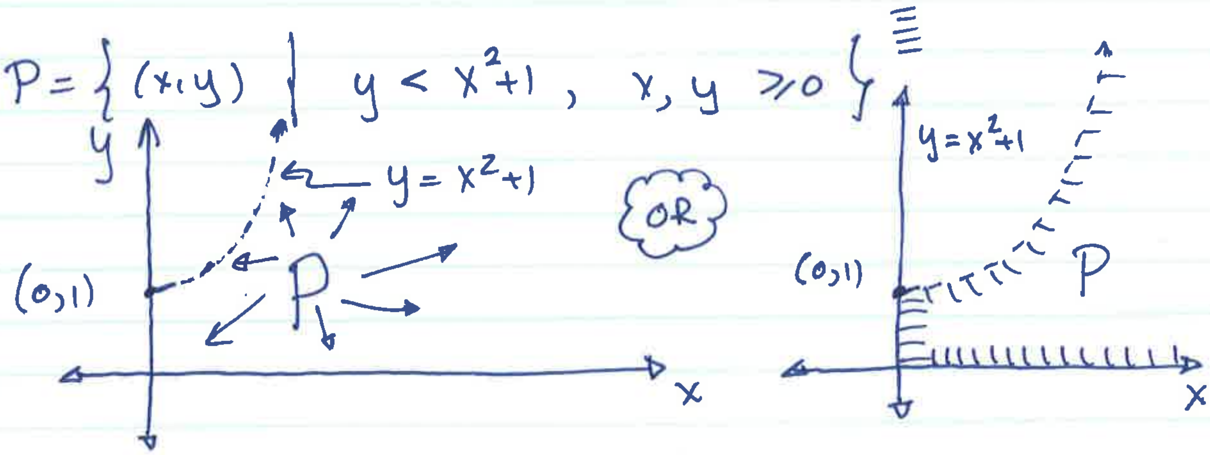
$\therefore$ we have that $y = -x - 1$ as required.

(d) Now we assume $f(x,y) = x^2 - y + 1$ where $x \geq 0$ is the number of units of X sold and $y \geq 0$ is the number of units of Y sold.

$$P = \{(x,y) \mid f(x,y) > 0,\ x \geq 0,\ y \geq 0\}$$

$L(0)$ was $y = x^2 + 1$. We want $f(x,y) > 0$ which gives $x^2 - y + 1 > 0$ so $y < x^2 + 1$

$$\therefore P = \{(x,y) \mid y < x^2 + 1,\ x, y \geq 0\}$$



Lastly, we consider $f_x(x,y) = f_y(x,y) = 0$ (*)

$f_x(x,y) = 2x \qquad f_y(x,y) = -1$

$\therefore$ we <u>never</u> have $f_x(x,y) = 0$ <u>and</u> $f_y(x,y) = 0$.

$\therefore$ there are <u>no</u> points in P satisfying (*). ∎

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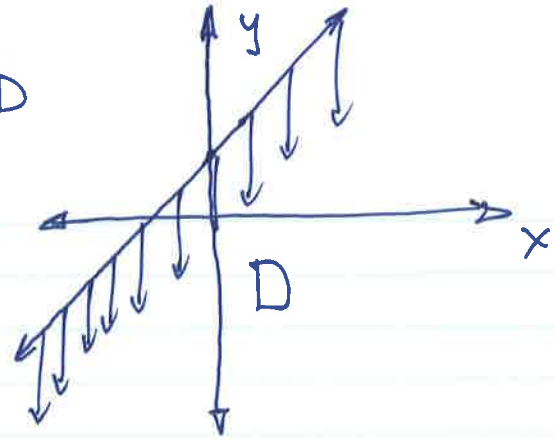
(5) Let $u(x,y) = z = \sqrt{2x - y + 1}$

(a) The domain of u is $\{(x,y) \mid 2x - y + 1 \geq 0\} = D$

$$\therefore D = \{(x,y) \mid y \leq 2x + 1\}$$

Here's a sketch of D
(D is shaded)



(b) $L(c) = \{(x,y) \mid u(x,y) = c\} = \{(x,y) \mid \sqrt{2x-y+1} = c\}$

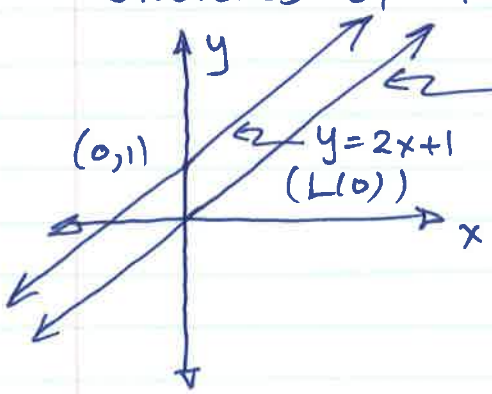
$L(0)$ gives $2x-y+1 = 0$ so $y = 2x+1$

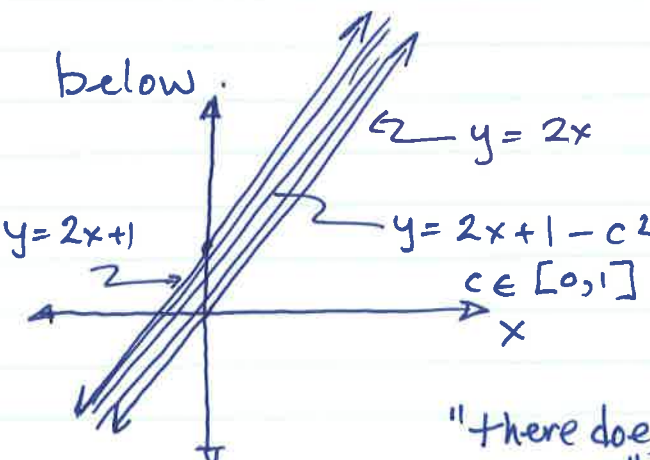
$L(1)$ gives $2x-y+1 = 1$ so $y = 2x$

If $c \in [0,1]$ then $\sqrt{2x-y+1} = c$ gives

$2x-y+1 = c^2$ so $y = 2x+1-c^2$

Sketches of these are below.





"there does NOT exist"

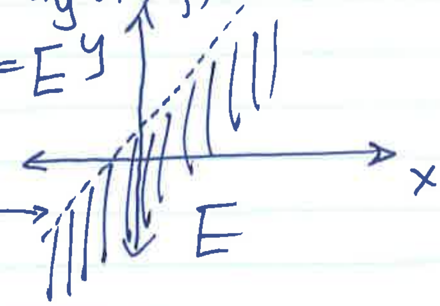
Lastly, suppose $c < 0$. For $L(c)$ we consider $u(x,y) = c$, so $\sqrt{2x-y+1} = c$. But $\nexists (x,y) \ni:$ $\sqrt{2x-y+1} = c$ if $c < 0$ because $\sqrt{2x-y+1} \geq 0$ $\forall (x,y) \in D$. $\therefore L(c) = \emptyset$ if $c < 0$.

"For All"   "empty set"

(c) $u_x(x,y) = \frac{1}{2}(2x-y+1)^{-1/2}(2) = \dfrac{1}{\sqrt{2x-y+1}}$

$u_y(x,y) = \frac{1}{2}(2x-y+1)^{-1/2}(-1) = \dfrac{-1}{2\sqrt{2x-y+1}}$

The domains of $u_x(x,y)$ and $u_y(x,y)$ are <u>both</u> $\{(x,y) \mid 2x-y+1>0\} = E$



We do <u>not</u> include the line $y = 2x+1$

$\left(u_x(x,y) + u_y(x,y)\right) u(x,y)$

$= \left(\dfrac{1}{\sqrt{2x-y+1}} - \dfrac{1}{2\sqrt{2x-y+1}}\right)\sqrt{2x-y+1}$

$= \dfrac{1}{2\sqrt{2x-y+1}}\sqrt{2x-y+1} = \underbrace{\frac{1}{2}}_{\text{constant}}$ for all $(x,y) \in E$. ■

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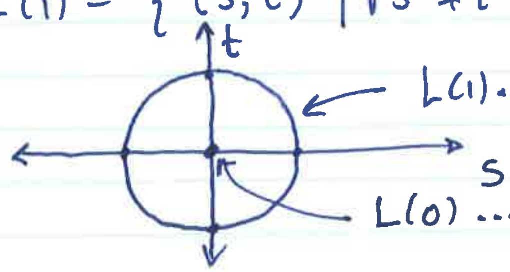
⑥ Let $z = F(s,t) = \sqrt{s^2+t^2}$

(a) $L(0) = \{(s,t) \mid \sqrt{s^2+t^2} = 0\} = \{(0,0)\}$ ... the origin.

$L(1) = \{(s,t) \mid \sqrt{s^2+t^2} = 1\} = \{(s,t) \mid \underbrace{s^2+t^2=1}_{\text{unit circle}}\}$

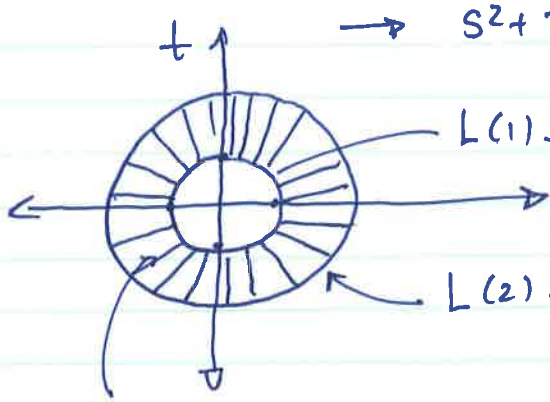


$L(1)$ .... $s^2+t^2=1$

$L(0)$ .... $\{(0,0)\}$

For $L(c)$ where $c \in [1,2]$, we consider

$$F(s,t) = c \longrightarrow \sqrt{s^2+t^2} = c$$
$$\longrightarrow s^2+t^2 = c^2 \quad \ldots \text{ circle center } (0,0), \text{ radius } c.$$



$L(c)$ where c ranges from 1 to 2 ... a "doughnut".

(b) $z_s = \dfrac{\partial z}{\partial s} = \dfrac{1}{2}(s^2+t^2)^{-1/2}(2s) = \dfrac{s}{\sqrt{s^2+t^2}}$

$z_t = \dfrac{\partial z}{\partial t} = \dfrac{1}{2}(s^2+t^2)^{-1/2}(2t) = \dfrac{t}{\sqrt{s^2+t^2}}$

$\therefore\ s z_s + t z_t = \dfrac{s^2}{\sqrt{s^2+t^2}} + \dfrac{t^2}{\sqrt{s^2+t^2}} = \sqrt{s^2+t^2}$

$= z \quad (\text{if } (s,t) \neq (0,0))$

(This is another example of a "partial differential equation")

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(7) (a) $z = x^y \qquad z_x = y x^{y-1} \qquad z_y = x^y \ln(x)$ (only if $x>0$)

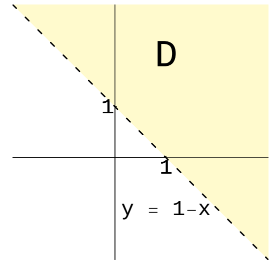
(b) $z = \log_x(y) = \dfrac{\ln(y)}{\ln(x)} = \ln(y)[\ln(x)]^{-1}$

$z_x = -\ln(y)[\ln(x)]^{-2}\dfrac{1}{x} = \dfrac{-\ln(y)}{x[\ln(x)]^2} = -\dfrac{\log_x(y)}{x\ln(x)}$

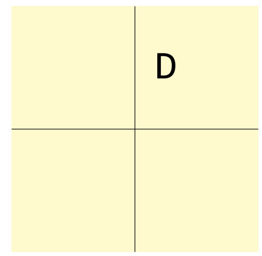
$z_y = \dfrac{1}{y\ln(x)}$

 # 9.

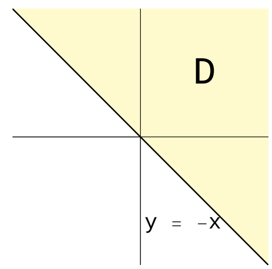
(a) Since $\ln t$ is only defined for $t > 0$, $f(x, y) = \ln(x + y - 1)$ is only defined if $x + y - 1 > 0$ or $y > 1 - x$. Hence the domain is $D = \{(x, y) \in \mathbb{R}^2 \mid y > 1 - x\}$. This is the region of the plane above the line $y = 1 - x$, excluding the line.



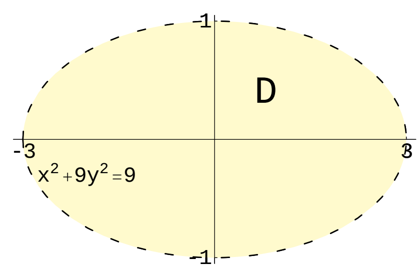
(b) $f(x, y) = e^{3\,x\,y}$ is defined for all pairs (x, y) of real numbers.



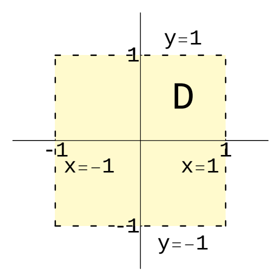
(c) Since $\sqrt{t}$ is only defined for $t \geq 0$, $f(x, y) = \sqrt{x + y}$ is defined for $y \geq -x$. Hence the domain is $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq -x\}$. This is the region of the plane above the line $y = -x$, including the line.



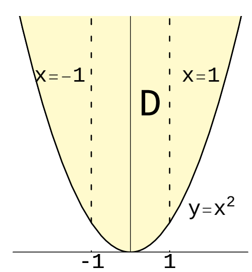
(d) Since $\ln t$ is only defined for $t > 0$, $f(x, y) = \ln(9 - x^2 - 9y^2)$ is only define if $9 - x^2 - 9y^2 > 0$ or $x^2 + 9y^2 < 9$. Hence the domain is $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 9y^2 < 9\}$. This is the region of the plane inside the ellipse $x^2 + 9y^2 = 9$, but excluding the ellipse.



(e) Since the domain of $\sqrt{1 - x^2}$ is $-1 < x < 1$ and the domain of $\sqrt{1 - y^2}$ is $-1 < y < 1$, the domain of $f(x, y) = \sqrt{1 - x^2} - \sqrt{1 - y^2}$ is $D = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, \; -1 < y < 1\}$. This is the part of the plane inside the square $[-1, 1] \times [-1, 1]$, but excluding the boundary lines.

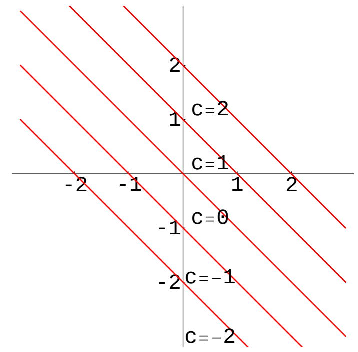


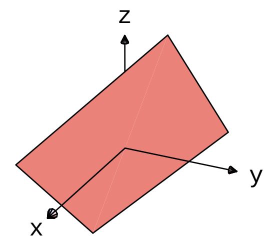
(f) We first note that $\sqrt{y - x^2}$ is only defined if $y - x^2 \geq 0$ or $y \geq x^2$ – the region of the plane above the graph of $y = x^2$, including the curve $y = x^2$. Also $\dfrac{1}{1 - x^2}$ is only define if $1 - x^2 \neq 0$ or $x \neq \pm 1$. hence the domain of $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$ is $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2, \; x \neq \pm 1\}$. This is the region of the plane above the graph of $y = x^2$, including the curve $y = x^2$, but excluding the lines $x = -1$ and $x = 1$.



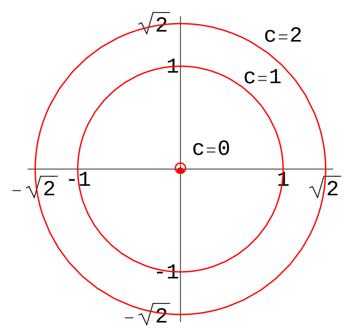
 # 10.

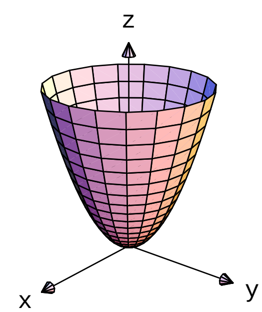
(a) The level curves are parallel lines. The graph of the function is a plane with xy–trace $y = -x$.



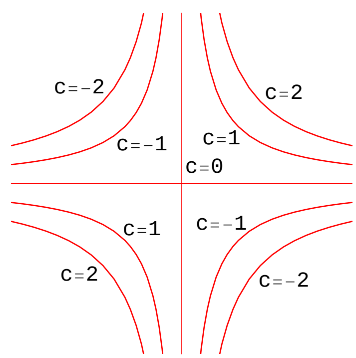


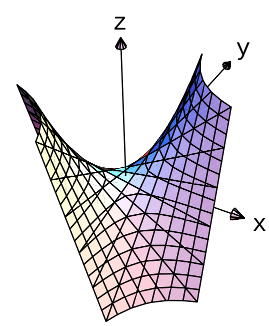
(b) The level curves are circles of radius $\sqrt{c}$ at the origin. Since $x^2 + y^2$ can never be negative, we can not draw level curves for $c = -2$ and $c = -1$. When $c = 0$, the level curve is only a point. The graph of the function is a paraboloid (bowl) opening upward with vertex at the origin.



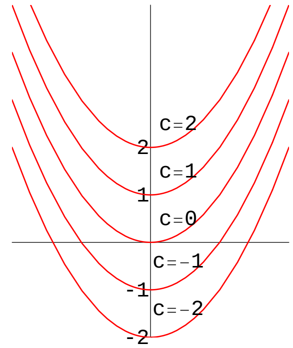


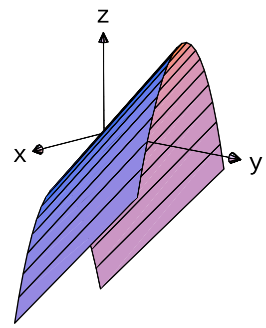
(c) The level curve for each c, $c \neq 0$ is a pair of hyperbolas in opposite quadrants of the plane. When $c = 0$, we get the x– axis and the y– axis. The graph of the function is saddle shaped.





(d) The level curves are parabolas. The graph of the function has xy–trace the parabola $y = x^2$. Hence it is a parabolic cylinder.





(e) The level curves are ellipses centered at the origin. Since $x^2 + 2y^2$ can never be negative, we can not draw level curves for $c = -2$ and $c = -1$. When $c = 0$, the level curve is only a point. The graph of the function is an ellipitical paraboloid opening upward with vertex at the origin.