

# SOLUTIONS ASSIGNMENT #4 MATA33

## ① Section 6.6 Problems

#2  $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

There is no need to reduce further.

$$[A | I] = \left[ \begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array} \right] R_2 - \frac{3}{2}R_1 \rightarrow R_2 \left[ \begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} & 1 \end{array} \right]$$

$A$  is not invertible because the row of 0's in the left submatrix means that  $A$  is not equivalent to  $I$

#6  $A = \begin{bmatrix} 2 & 0 & 8 \\ -1 & 4 & 0 \\ 2 & 1 & 0 \end{bmatrix} \quad \frac{1}{2}R_1 \rightarrow R_1 \quad [A | I]$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 + R_1 \rightarrow R_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array} \right]$$

$$R_3 - 2R_1 \rightarrow R_3$$

$$\left( \frac{1}{4} \right) R_2 \rightarrow R_2 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array} \right]$$

$\therefore A$  is reducible to  $I$ , the matrix  $A$  is invertible, and

$$A^{-1} = \begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix}$$

$$R_3 - R_2 \rightarrow R_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array} \right]$$

$$\left( -\frac{1}{9} \right) R_3 \rightarrow R_3 \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array} \right]$$

$$R_1 - 4R_3 \rightarrow R_1$$

$$R_2 - R_3 \rightarrow R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array} \right]$$

$I$ 
 $A^{-1}$

(2)

$$\#8 \ [A | I] = \left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{aligned} (\frac{1}{2})R_1 \rightarrow R_1 \\ -\frac{1}{4}R_3 \rightarrow R_3 \end{aligned} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array} \right]$$

$$R_2 \leftrightarrow R_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

The reduced form of  $A$  is not  $I$ , so  $A$  is not invertible. ▣

$$\#18 \ [A | I] = \left[ \begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{aligned} (\frac{1}{2})R_2 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{aligned} \left[ \begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$(\frac{1}{2})R_1 \rightarrow R_1 \left[ \begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{aligned} R_1 + \frac{1}{2}R_2 \rightarrow R_1 \\ R_3 - 2R_2 \rightarrow R_3 \end{aligned} \left[ \begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array} \right]$$

$$(-\frac{1}{2})R_3 \rightarrow R_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$R_1 - \frac{3}{2} R_3 \rightarrow R_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

(3)

$\therefore A$  reduces to  $I$ ,  $A^{-1}$   $\exists$ , and  $A^{-1} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

↑ "exists"



#20 For the matrix equation  $AX = B$  we're given that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From  $AX = B$  we have  $X = A^{-1}B$

$$\therefore X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 16 \end{bmatrix}$$

$\therefore X = 9, y = 6, z = 16$  is the solution



#22 Given linear system is  $2x + 4y = 5$   
 $-x + 3y = -2$

Using the inverse matrix approach:

$$\left[ \begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ -1 & 3 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 5 & \frac{1}{2} & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{10} & \frac{1}{5} \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{10} & \frac{1}{5} \end{array} \right] \quad \left( \begin{array}{l} \text{You} \\ \text{supply the} \\ \text{ERO notation} \end{array} \right)$$

I  $A^{-1}$

$$\text{For } AX=B, X=A^{-1}B = \begin{bmatrix} \frac{3}{10} & -\frac{2}{5} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad (4)$$

$$\therefore X = \frac{23}{10}, y = \frac{1}{10} \text{ is the solution.} \quad = \begin{bmatrix} \frac{23}{10} \\ \frac{1}{10} \end{bmatrix}$$



#28 Given linear system is 
$$\left. \begin{array}{l} x + y + z = 6 \\ x - y + z = -1 \\ x - y - z = 4 \end{array} \right\} (*)$$

We first try the matrix-inverse approach:

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$(-\frac{1}{2})R_2 \rightarrow R_2 \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 - R_2 \rightarrow R_1 \\ R_3 + 2R_2 \rightarrow R_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array} \right]$$

$$(-\frac{1}{2})R_3 \rightarrow R_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$R_1 - R_3 \rightarrow R_1 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

I
A<sup>-1</sup>

In matrix form, the system (\*) is  $AX = B$

(5)

$$\therefore X = A^{-1}B = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \\ -\frac{5}{2} \end{bmatrix}$$

$$\therefore x = 5, \quad y = \frac{7}{2}, \quad z = -\frac{5}{2}$$



#32 Given system is 
$$\left. \begin{array}{l} x + 3y + 3z = 7 \\ 2x + y + z = 4 \\ x + y + z = 3 \end{array} \right\} (*)$$

We consider first the inverse-matrix approach:

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array} \right]$$

$$R_3 - \frac{2}{5}R_2 \rightarrow R_3 \left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & 1 \end{array} \right] \text{ The row of 0's in the left submatrix shows that the coefficient matrix for (*) is not invertible.}$$

We now use the method of reduction to solve (\*)

$$\left[ \begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -4 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{array} \right] \left(-\frac{1}{5}\right)R_2 \rightarrow R_2 \left[ \begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_1 - 3R_2 \rightarrow R_1 \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \therefore \text{the solution to (*) is } x = 1, \quad y = -r + 2, \quad z = r$$

where  $r \in \mathbb{R}$  is arbitrary.



#36  $A = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}$   $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -4 & -2 \end{bmatrix}$  (6)

We find  $(I - A)^{-1}$ :

$$\left[ \begin{array}{cc|cc} 4 & -2 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array} \right] \left(\frac{1}{4}\right)R_1 \rightarrow R_1 \left[ \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ -4 & -2 & 0 & 1 \end{array} \right]$$

$$R_2 + 4R_1 \rightarrow R_2 \left[ \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -4 & 1 & 1 \end{array} \right] \left(-\frac{1}{4}\right)R_2 \rightarrow R_2 \left[ \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array} \right]$$

$$R_1 + \frac{1}{2}R_2 \rightarrow R_1 \left[ \begin{array}{cc|cc} 1 & 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array} \right]$$

$\underbrace{\hspace{10em}}_I \quad \underbrace{\hspace{10em}}_{(I-A)^{-1}}$



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#40  $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$   $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

For  $A^{-1}$ ,  $\left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array} \right]$

$\therefore A^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$  so  $(A^{-1})^T = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

For  $(A^T)^{-1}$ ,  $\left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right]$

$\rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right] \therefore (A^T)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

$\therefore$  we have  $(A^{-1})^T = (A^T)^{-1}$  (as asserted in the question)

#42  $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ -1 & -2 & 2 \end{bmatrix}$   $R_1 = [33 \ 87 \ 70]$   
 $R_2 = [57 \ 133 \ 20]$   
 $R_3 = [38 \ 90 \ 33]$

(a) Use the procedure for finding the inverse of a matrix (p. 281) to show how one gets

$$A^{-1} = \begin{bmatrix} 14 & -2 & 9 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore R_1 A^{-1} = [10 \ 21 \ 19] = [J \ U \ S]$$

$$R_2 A^{-1} = [20 \ 19 \ 1] = [T \ S \ A]$$

$$R_3 A^{-1} = [25 \ 14 \ 15] = [Y \ N \ O]$$

(b) Message is "JUST SAY NO". ▣

② Page 292 #32 Given is an  $n \times n$  invertible matrix  $A$ .

(a) Show that  $A^k$  is invertible for any  $k \in \mathbb{N}$  (We use the power as "k" rather than "n" because n is the size (i.e. order) of  $A$ ).

Use the definition on page 278:

$$\text{Let } C = (A^{-1})^k = \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ products}}$$

$$\begin{aligned} \text{Then } CA^k &= (A^{-1} A^{-1} \cdots A^{-1})(A A A \cdots A) \\ &= \underbrace{(A^{-1} A^{-1} \cdots A^{-1})}_{k-1} (A^{-1} A) \underbrace{(A \cdots A)}_{k-1} \end{aligned}$$

$$= (A^{-1} \dots A^{-1}) \underbrace{(A^{-1}A)}_I \underbrace{(A \dots A)}_{k-2} \text{ (and so forth) } \textcircled{8}$$

$$\dots = A^{-1}A = I$$

It follows from the definition of invertibility on page 278 that  $A$  is invertible and  $C = (A^{-1})^k$  is the inverse of  $A^k$ .

(b) Assume  $B$  &  $C$  are  $n \times n$  and  $ABA = ACA$  ( $A$  is as above)

$$\text{We have } A^{-1}(AB)A = A^{-1}(AC)A$$

$$\Rightarrow (A^{-1}A)(BA) = (A^{-1}A)(CA)$$

$$\Rightarrow BA = CA$$

$$\Rightarrow (BA)A^{-1} = (CA)A^{-1} \text{ so } B = C.$$

(c) Suppose  $A^2 = A$ .  $\therefore A^{-1}(A^2) = A^{-1}A$

(Where  $A$  is as above.....

$$(A^{-1}A)A = I$$

$n \times n$ , invertible)

$$IA = I \Rightarrow A = I$$



③ We reduce  $A$ :

$$A = \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \xrightarrow{R_1 - R_2 \rightarrow R_1} \begin{array}{ccccc|c} 1 & 6 & 0 & 3 & -1 & -7 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}$$

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 6 & 0 & 3 & -1 & -7 \\ 0 & 0 & 1 & -4 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \xrightarrow{\begin{array}{l} R_1 + R_3 \rightarrow R_1 \\ R_2 + \frac{1}{2}R_3 \rightarrow R_2 \end{array}} \begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}$$

(Matrix is reduced)

If  $x_1, \dots, x_5$  are the variables for the corresponding system, then  $x_1 = -6r - 3s$   $x_3 = 4s + 5$

$x_2 = r$   $x_4 = s$   $x_5 = 7$   
 $r, s$  are parameters;  $r, s \in \mathbb{R}$ .



④ Given the linear system 
$$\begin{cases} x + 3y + 4z = a \\ -4x + 2y - 6z = b \\ -3x - 2y - 7z = c \end{cases} \quad (*)$$

where  $a, b, c$  are constants, we ask whether the system has a solution for each choice of  $a, b, c$ ?

To answer this, reduce the augmented matrix for (\*):

$$\left[ \begin{array}{ccc|c} 1 & 3 & 4 & a \\ -4 & 2 & -6 & b \\ -3 & -2 & -7 & c \end{array} \right] \begin{array}{l} R_2 + 4R_1 \rightarrow R_2 \\ R_3 + 3R_1 \rightarrow R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & 3 & 4 & a \\ 0 & 14 & 10 & b + 4a \\ 0 & 7 & 5 & c + 3a \end{array} \right]$$

$$\left(\frac{1}{14}\right) R_2 \rightarrow R_2 \left[ \begin{array}{ccc|c} 1 & 3 & 4 & a \\ 0 & 1 & \frac{5}{7} & \frac{b}{14} + \frac{2}{7}a \\ 0 & 7 & 5 & c + 3a \end{array} \right]$$

$$R_3 - 7R_2 \rightarrow R_3 \left[ \begin{array}{ccc|c} 1 & 3 & 4 & a \\ 0 & 1 & \frac{5}{7} & \frac{b}{14} + \frac{2}{7}a \\ 0 & 0 & 0 & a - \frac{1}{2}b + c \end{array} \right] \text{ We can stop here.}$$

The row of 0's in row 3 means that the system (\*) has a solution if and only if  $a, b, c$  satisfy the equation  $a - \frac{1}{2}b + c$

(This is because the 3<sup>rd</sup> row corresponds to the equation  $0x + 0y + 0z = a - \frac{1}{2}b + c$ , which has a solution only when  $a - \frac{1}{2}b + c = 0$ )

$\therefore$  the answer to the stated question is no — (\*) only has a solution if  $a - \frac{1}{2}b + c = 0$ .



⑤ If  $A$  &  $B$  are  $2 \times 2$  invertible matrices, is  $A+B$  also invertible?

⑩

Answer: Generally, No! Here is a compelling counter example. The  $2 \times 2$  zero matrix  $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is not invertible. Let  $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$  and  $B = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix}$  where  $a \in \mathbb{R}$ ,  $a \neq 0$ . Then  $A$  and  $B$  are invertible ( $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$ ,  $B^{-1} = \begin{bmatrix} -\frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}$ ) but  $A+B=O$ .

(It is sometimes the case that  $A$  &  $B$  invertible imply  $A+B$  is invertible (e.g.  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$   $A+B = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ ) but it is not a general principle, and therefore not a theorem) 

⑥ Assume  $A$  is  $3 \times 3$  and  $A^4 - 2A^3 + 5A^2 - 2I = 0$  (\*)

We show that  $A$  is invertible.

Solution: Factor (\*) and write  $AC = I$

$$\text{From (*)}, \quad A^4 - 2A^3 + 5A^2 = 2I$$

$$\therefore \frac{1}{2}A^4 - A^3 + \frac{5}{2}A^2 = I$$

$$\therefore A\left(\frac{1}{2}A^3 - A^2 + \frac{5}{2}A\right) = I$$

Let  $C = \frac{1}{2}A^3 - A^2 + \frac{5}{2}A$ , so  $C$  is  $3 \times 3$  and  $A$  is invertible (Reference is the definition, p278) 

⑦  $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$   $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$   $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(a)  $P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$  so we get

$$PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} = A.$$

(b)  $A^3 = (PDP^{-1})^3 = PDP^{-1}PDP^{-1}PDP^{-1}$  (Repeated use of  $P^{-1}P = I$ )

$$= PD^3P^{-1}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^3 & 0 \\ 0 & 1^3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 22 & -21 \\ 14 & -13 \end{bmatrix}$$

$A^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1}$  (You should check that this is correct)

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3070 & -3069 \\ 2046 & -2045 \end{bmatrix}$$

(c)  $A^n = (PDP^{-1})^n = PD^nP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & -2^n \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3(2^n) - 2 & -3(2^n) + 3 \\ 2^{n+1} - 2 & -2^{n+1} + 3 \end{bmatrix}$$

$$8) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & x & 2 \\ 4 & 1 & x+2 & 3 \end{bmatrix} \begin{array}{l} R_2 - 2R_1 \\ \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array} = \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & x-9 & -10 \\ 0 & -7 & x-10 & -13 \end{bmatrix}$$

$$= \det \begin{bmatrix} -1 & -2 & -7 \\ -2 & x-9 & -10 \\ -7 & x-10 & -13 \end{bmatrix} = - \det \begin{bmatrix} 1 & 2 & 7 \\ -2 & x-9 & -10 \\ -7 & x-10 & -13 \end{bmatrix}$$

$$\begin{array}{l} R_3 + 7R_1 \\ R_2 + 2R_1 \end{array} = - \det \begin{bmatrix} 1 & 2 & 7 \\ 0 & x-5 & 4 \\ 0 & x+4 & 36 \end{bmatrix} = - \det \begin{bmatrix} x-5 & 4 \\ x+4 & 36 \end{bmatrix} = - \det \begin{bmatrix} x-5 & 4 \\ 9 & 32 \end{bmatrix}$$

$$= - (32x - 160 - 36) = - (32x - 196).$$

$$A \text{ is not invertible} \Leftrightarrow \det A = 0 \Leftrightarrow x = \frac{196}{32} = \frac{49}{8}.$$