

**University of Toronto Scarborough**  
**Department of Computer & Mathematical Sciences**

**MAT A33H**

**Winter 2011**

Solutions #11

**Important:** If the region of integration is described in words or, if you are asked to reverse the order of integration when the “inside” integral involves a variable, you should first sketch the region of integration and label its boundaries. It also helps to include in the sketch the vertical and/or the horizontal bar as appropriate. You do not need to sketch the region when all the limits of integration are numbers.

**Section 17.9 (12ed. 17.10)**

$$6. \int_{-2}^3 \int_0^2 (y^2 - 2xy) dy dx = \int_{-2}^3 \left[ \frac{y^3}{3} - xy^2 \right]_0^2 dx = \int_{-2}^3 \left( \frac{8}{3} - 4x \right) dx = \left[ \frac{8x}{3} - 2x^2 \right]_{-2}^3 = \frac{10}{3}.$$

$$8. \int_0^3 \int_0^x (x^2 + y^2) dy dx = \int_0^3 \left[ x^2 y + \frac{y^3}{3} \right]_0^x dx = \int_0^3 \left( x^3 + \frac{x^3}{3} \right) dx = \int_0^3 \frac{4x^3}{3} dx = \left. \frac{x^4}{3} \right|_0^3 = 27.$$

$$10. \int_1^2 \int_0^{x-1} 2y dy dx = \int_1^2 \left[ y^2 \right]_0^{x-1} dx = \int_1^2 (x-1)^2 dx = \left. \frac{(x-1)^3}{3} \right|_1^2 = \frac{1}{3}.$$

$$12. \int_0^2 \int_0^{x^2} xy dy dx = \int_0^2 \left[ \frac{xy^2}{2} \right]_0^{x^2} dx = \int_0^2 \frac{x^5}{2} dx = \left. \frac{x^6}{12} \right|_0^2 = \frac{16}{3}.$$

$$14. \int_0^1 \int_{y^2}^y y dx dy = \int_0^1 \left[ xy \right]_{y^2}^y dy = \int_0^1 (y^2 - y^3) dy = \left[ \frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$16. \int_0^3 \int_{y^2}^{3y} 5x dx dy = \int_0^3 \left[ \frac{5x^2}{2} \right]_{y^2}^{3y} dy = \int_0^3 \left( \frac{45y^2}{2} - \frac{5y^4}{2} \right) dy = \left[ \frac{15y^3}{2} - \frac{y^5}{2} \right]_0^3 = \frac{405}{2} - \frac{243}{2} = 81.$$

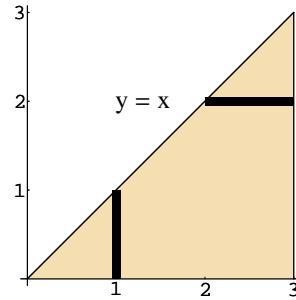
$$18. \int_0^1 \int_0^1 e^{y-x} dx dy = \int_0^1 \left[ -e^{y-x} \right]_0^1 dy = \int_0^1 (-e^{y-1} + e^y) dy = \left[ -e^{y-1} + e^y \right]_0^1 = -e^0 + e^1 + e^{-1} - e^0 = e - 2 + e^{-1}.$$

$$20. \int_0^1 \int_0^x \int_0^{x+y} x^2 dz dy dx = \int_0^1 \int_0^x \left[ x^2 z \right]_0^{x+y} dy dx = \int_0^1 \int_0^x x^2(x+y) dy dx = \int_0^1 \int_0^x (x^3 + x^2 y) dy dx = \int_0^1 \int_0^x \left[ x^3 y + \frac{x^2 y^2}{2} \right]_0^x dx = \int_0^1 \left( x^4 + \frac{x^4}{2} \right) dx = \int_0^1 \frac{3x^4}{2} dx = \left. \frac{3x^5}{10} \right|_0^1 = \frac{3}{10}.$$

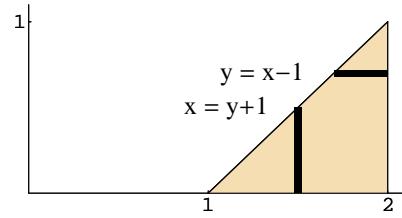
$$22. \int_1^e \int_{\ln x}^x \int_0^y dz dy dx = \int_1^e \int_{\ln x}^x \left[ z \right]_0^y dy dx = \int_0^y \int_{\ln x}^x y dy dx = \int_1^e \left[ \frac{y^2}{2} \right]_{\ln x}^x dx = \int_1^e \left( \frac{x^2}{2} - \frac{(\ln x)^2}{2} \right) dx = \left[ \frac{x^3}{6} - \frac{1}{2}(x \ln^2 x - 2x \ln x + 2x) \right]_1^e = \frac{e^3}{6} - \frac{e}{2} + \frac{5}{6}.$$

# 2 Redo problems 8 – 18 from the previous question by reversing the order of integration.

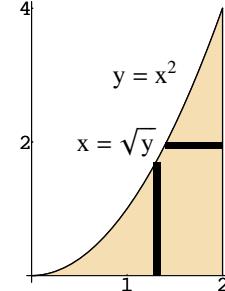
$$8. \int_0^3 \int_0^x (x^2 + y^2) dy dx = \int_0^3 \int_y^3 (x^2 + y^2) dx dy = \int_0^3 \left[ \frac{x^3}{3} + xy^2 \right]_{x=y}^{x=3} dy = \int_0^3 \left( 9 + 3y^2 - \frac{y^3}{3} - y^3 \right) dy = \int_0^3 \left( 9 + 3y^2 - \frac{4y^3}{3} \right) dy = \left[ 9y + y^3 - \frac{y^4}{3} \right]_0^3 = 27.$$



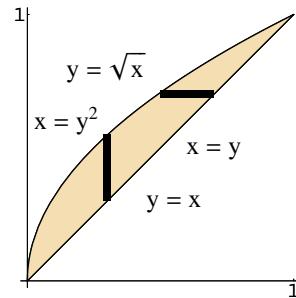
$$10. \text{ Since } y = x - 1 \text{ can be rewritten as } x = y + 1, \text{ we have } \int_1^2 \int_0^{x-1} 2y dy dx = \int_0^1 \int_{y+1}^2 2y dx dy = \int_0^1 \left[ 2yx \right]_{x=y+1}^{x=2} dy = 2 \int_0^1 (2y - y(y+1)) dy = 2 \int_0^1 (y - y^2) dy = 2 \left[ \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}.$$



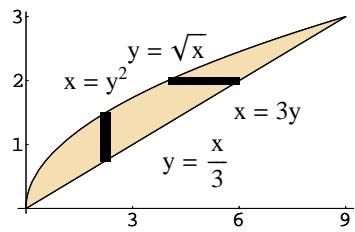
$$12. \text{ We first note that, in the first quadrant, the curve } y = x^2 \text{ can also be given by } x = \sqrt{y}. \text{ Hence } \int_0^2 \int_0^{x^2} xy dy dx = \int_0^4 \int_{\sqrt{y}}^2 xy dx dy = \int_0^4 \left[ \frac{x^2 y}{2} \right]_{x=\sqrt{y}}^{x=2} dy = \int_0^4 \left( 2y - \frac{y^2}{2} \right) dy = \left[ y^2 - \frac{y^3}{6} \right]_0^4 = 16 - \frac{64}{6} = \frac{16}{3}.$$



$$14. \text{ We first note that, in the first quadrant, the curve } x = y^2 \text{ can also be given by } y = \sqrt{x}. \text{ Hence } \int_0^1 \int_{y^2}^y y dx dy = \int_0^1 \int_x^{\sqrt{x}} y dx dy = \int_0^1 \left[ \frac{y^2}{2} \right]_{y=x}^{y=\sqrt{x}} dx = \int_0^1 \left( \frac{x}{2} - \frac{x^2}{2} \right) dx = \left[ \frac{x^2}{4} - \frac{x^3}{6} \right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$



$$16. \text{ Giving alternate descriptions for the boundaries we have } y = \sqrt{x} \text{ for } x = y^2 \text{ and } y = \frac{x}{3} \text{ for } x = 3y. \text{ Now } \int_0^3 \int_{y^2}^{3y} 5x dx dy = \int_0^9 \int_{\frac{y}{3}}^{\sqrt{x}} 5x dy dx = \int_0^9 \left[ 5xy \right]_{y=\frac{y}{3}}^{y=\sqrt{x}} dx = 5 \int_0^9 \left( x^{\frac{3}{2}} - \frac{x^2}{3} \right) dx = 5 \left[ \frac{2x^{\frac{5}{2}}}{5} - \frac{x^3}{9} \right]_0^9 = 5 \left( \frac{486}{5} - 81 \right) = 81.$$



$$18. \text{ This time the limits of integration are just numbers. Hence } \int_0^1 \int_0^1 e^{y-x} dx dy = \int_0^1 \int_0^1 e^{y-x} dy dx = \int_0^1 \left[ e^{y-x} \right]_{y=0}^{y=1} dx = \int_0^1 (e^{1-x} - e^{-x}) dx = \left[ -e^{1-x} + e^{-x} \right]_0^1 = -2 + e + e^{-1}.$$

# 3

$$(a) \int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3} dx dy.$$

Since the integral of  $\sqrt{2+x^3}$  can not be computed as an elementary function, we need to reverse the order of integration.  $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3} dx dy = \int_0^1 \int_0^{x^2} \sqrt{2+x^3} dy dx = \int_0^1 x^2 \sqrt{2+x^3} dx = \left[ \frac{2}{9} (2+x^3)^{3/2} \right]_0^1 = \frac{2}{9} (3\sqrt{3} - 2\sqrt{2}).$

$$(b) \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy dx.$$

Since the integral of  $e^{y^3}$  w.r.t.  $y$  can not be computed as an elementary function, we need to reverse the order of integration.

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy dx = \int_0^1 \int_0^{y^2} e^{y^3} dx dy = \int_0^1 e^{y^3} \left[ x \right]_0^{y^2} dy = \int_0^1 y^2 e^{y^3} dy = \left[ \frac{1}{3} e^{y^3} \right]_0^1 = \frac{e-1}{3}.$$

$$(c) \int_0^1 \int_{e^y}^e \frac{x}{\ln x} dx dy.$$

The region  $e^y \leq x \leq e$ ,  $0 \leq y \leq 1$  can also be written as  $0 \leq y \leq \ln x$ ,  $1 \leq x \leq e$ . So, changing the order of integration we have

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} dx dy = \int_1^e \int_0^{\ln x} \frac{x}{\ln x} dy dx = \int_1^e \left[ y \frac{x}{\ln x} \right]_0^{\ln x} dx = \int_1^e x dx = \left[ \frac{x^2}{2} \right]_1^e = \frac{1}{2} (e^2 - 1).$$

$$(d) \iint_D f(x, y) dA, \text{ where } f(x, y) = y - x, \text{ and } D \text{ is the triangle with vertices } (0, 0), (1, 0) \text{ and } (2, 1).$$

$$\begin{aligned} \int_D f(x, y) dA &= \int_0^1 \int_{2y}^{y+1} (y-x) dx dy = \int_0^1 \left[ yx - \frac{x^2}{2} \right]_{2y}^{y+1} dy = \\ &= \int_0^1 \left( y(y-1) - \frac{(y+1)^2}{2} - 2y^2 + 2y^2 \right) dy = \frac{1}{2} \int_0^1 (y^2 - 1) dy = \\ &= \frac{1}{2} \left[ \frac{y^3}{3} - y \right]_0^1 = -\frac{1}{3}. \end{aligned}$$

$$(e) \int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y} dy dx.$$

We can not integrate  $e^{x/y}$  w.r.t.  $y$  by exact means, so we will reverse the order of integration.

$$\begin{aligned} \int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y} dy dx &= \int_0^1 \int_{y^3}^y e^{x/y} dx dy = \\ &\int_0^1 \left( ye^{x/y} \Big|_{y^3}^y \right) dy = \int_0^1 (ye - ye^{y^2}) dy = \left[ \frac{y^2 e}{2} - \frac{e^{y^2}}{2} \right]_0^1 = \frac{1}{2}. \end{aligned}$$

