

(1) Problems from Section 17.7

#2 Given function is $f(x,y) = 3x^2 - 2y^2 + 9$. and the constraint is $x+y=1$. We use Lagrange Multipliers (LM) to find critical point(s) (CP) of f satisfying the constraint.

Solution: Lagrangian is $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$$\text{where } g(x,y) = x+y+1$$

$$\therefore F = 3x^2 - 2y^2 + 9 - \lambda(x+y+1)$$

$$F_x = 6x - \lambda = 0 \rightarrow x = \frac{\lambda}{6} \quad (1)$$

$$F_y = -4y - \lambda = 0 \rightarrow y = -\frac{\lambda}{4} \quad (2)$$

$$F_\lambda = -x - y + 1 = 0 \quad (3)$$

$$\text{Sub (1) \& (2) into (3) to get } -\frac{\lambda}{6} + \frac{\lambda}{4} + 1 = 0 \text{ so } \lambda = -12$$

$$\therefore x = -2, y = 3$$

\therefore the only (constrained) CP of f is $(-2, 3)$ 

#4 Given function is $f(x,y,z) = x+y+z$ and the constraint is $xyz = 8$. We find the "constrained" critical point(s) of f (i.e. the CP(s) of f that satisfy the constraints)

Solution: Lagrangian is $F(x,y,z,\lambda) = f - \lambda g$

$$\text{where } g = g(x,y,z) = xyz - 8$$

$$F(x, y, z, \lambda) = x + y + z - \lambda(xy - 8) \quad (2)$$

$$\therefore F_x = 1 - \lambda yz = 0 \quad (1) \rightarrow \lambda, y, z \neq 0$$

$$F_y = 1 - \lambda xz = 0 \quad (2) \rightarrow x \neq 0$$

$$F_z = 1 - \lambda xy = 0 \quad (3)$$

$$F_\lambda = 0 \rightarrow xyz = 8 \quad (4)$$

$$(1) + (2) \rightarrow \lambda yz = \lambda xz \text{ so } y = x (\because \lambda, z \neq 0)$$

$$(2) + (3) \rightarrow \lambda xz = \lambda xy \text{ so } y = z (\because \lambda, x \neq 0)$$

$\therefore x = y = z$ and we substitute into (4) to get

$$x^3 = 8 \rightarrow x = y = z = 2 \therefore \text{the CP of } f$$

satisfying the constraint is $(2, 2, 2)$. 

#8 Given function is $f(x, y, z) = x^2 + y^2 + z^2$ and we have the constraint $x + y + z = 3$. Use LM to find the CP(s) of f satisfying the constraint.

Solution : Lagrangian is $F = f - \lambda g$ where

$$g = g(x, y, z) = x + y + z - 3$$

$$\therefore F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 3)$$

$$F_x = 2x - \lambda = 0 \quad (1) \quad (1), (2) + (3) \text{ imply that}$$

$$F_y = 2y - \lambda = 0 \quad (2) \quad x = y = z$$

$$F_z = 2z - \lambda = 0 \quad (3) \quad \therefore \text{substitution in (4)}$$

$$F_\lambda = 0 \rightarrow x + y + z = 3 \quad (4) \text{ gives } x = y = z = 1$$

\therefore the CP of f satisfying g is $(1, 1, 1)$. 

(3)

#14 We use LM to minimize the cost function
 $C = C(q_1, q_2) = 3q_1^2 + q_1 q_2 + 2q_2^2$ subject to
the constraint $q_1 + q_2 = 200$. Here q_1, q_2 are
the production levels at plants 1 and 2, respectively.

Solution: Lagrangian is $F = c - \lambda g$ where

$$g = g(q_1, q_2) = q_1 + q_2 - 200.$$

$$\therefore F(q_1, q_2, \lambda) = 3q_1^2 + q_1 q_2 + 2q_2^2 - \lambda(q_1 + q_2 - 200)$$

$$F_{q_1} = 6q_1 + q_2 - \lambda = 0 \quad (1) \quad (1) - (2): 5q_1 - 3q_2 = 0$$

$$F_{q_2} = q_1 + 4q_2 - \lambda = 0 \quad (2) \quad \therefore q_1 = \frac{3}{5}q_2$$

$$F_\lambda = 0 \rightarrow q_1 + q_2 = 200 \quad (3) \quad \text{Sub into (3): } \frac{8}{5}q_2 = 200$$

$$\therefore q_2 = 125 \text{ and } q_1 = 75$$

Assuming that the CP of c satisfying the constraint does minimize C , we conclude that C is minimized when plant 1 produces 75 units and plant 2 produces 125 units. 

#16 A production function is $f(l, k) = 20l + 25k - l^2 - 3k^2$
where a budget constraint is $2l + 4k = 50$

Assuming the CP(s) obtained via LM maximize output, find the maximizing output.

Solution: Lagrangian is $F = f - \lambda g$ where
 $g = g(l, k) = 2l + 4k - 50$.

$$\therefore F(l, k, \lambda) = 20l + 25k - l^2 - 3k^2 - \lambda(2l + 4k - 50) \quad (4)$$

$$F_l = 20 - 2l - 2\lambda = 0 \quad (1) \rightarrow l = 10 - \lambda$$

$$F_k = 25 - 6k - 4\lambda = 0 \quad (2) \rightarrow k = \frac{25}{6} - \frac{2}{3}\lambda$$

$$F_\lambda = 0 \rightarrow 2l + 4k - 50 = 0 \quad (3) \quad \text{Substitute in (3):}$$

$$2(10 - \lambda) + 4\left(\frac{25}{6} - \frac{2}{3}\lambda\right) - 50 = 0$$

$$\text{Algebra shows that } \lambda = -\frac{20}{7} \text{ so}$$

$l = \frac{90}{7}$ and $k = \frac{85}{14}$ \therefore the greatest output

(i.e. production) is $f\left(\frac{90}{7}, \frac{85}{14}\right) = \frac{3725}{28} \approx 133$

units when $l = \frac{90}{7}$ and $k = \frac{85}{14}$. 

#20 A production function is given by

$$q = q(l, k) = \frac{1}{16} [65 - 4(l-4)^2 - 2(k-5)^2]$$

and manufacturing cost is given by

$c = c(l, k) = 8l + 16k$. The selling price per unit is 64.

(a) Let P represent total profit as a function of l and k. We have

$$P = (\text{Total Revenue}) - (\text{Total Cost})$$

$$= 64q(l, k) - c(l, k)$$

$$= \frac{64}{16} [65 - 4(l-4)^2 - 2(k-5)^2] - 8l - 16k$$

(5)

$$\therefore P = -16l^2 + 120l - 8k^2 + 64k - 196$$

(by elementary algebra)

(b) We find CP(s) of P and analyze them via the 2nd D-test

$$P_l = -32l + 120 = 0 \rightarrow l = \frac{15}{4} \quad \therefore \text{the only CP of } P \text{ is}$$

$$P_k = -16k + 64 = 0 \rightarrow k = 4 \quad \left(\frac{15}{4}, 4\right)$$

$$D(l, k) = P_{ll} \cdot P_{kk} - (P_{lk})^2 = (-32)(-16) - 0 \\ = 512$$

$$\therefore D\left(\frac{15}{4}, 4\right) = 512 > 0 \quad P_{ll}\left(\frac{15}{4}, 4\right) = -32 < 0$$

\therefore by 2nd DT, P has a relative maximum

when $l = \frac{15}{4}$ & $k = 4$. The profit at this CP is $P\left(\frac{15}{4}, 4\right) = 157$

Important Remark $\because P$ is essentially a quadratic function separately in l and k , we can complete the square in P and determine that, at $\left(\frac{15}{4}, 4\right)$, we actually have an absolute maximum:

$$P = -16\left(l^2 - \frac{15}{2}l\right) - 8(k^2 - 8k) - 196$$

$$= -16\left(l - \frac{15}{4}\right)^2 - 8(k-4)^2 + 157$$

$$P\left(\frac{15}{4}, 4\right) = 157. \text{ If } (l, k) \neq \left(\frac{15}{4}, 4\right), P(l, k) < 157.$$

(6)

(C) Now we view profit as a function of 3 variables l, k, g :

$P = 64g - 8l - 16k$, where we have a constraint: $16g = 65 - 4(l-4)^2 - 2(k-5)^2$

Use the technique of LM to find the CP(s) of P satisfying the constraint.

Solution: Lagrangian is $F(l, k, g, \lambda) = P - \lambda g$

where $g = g(l, k, g) = 16g - 65 + 4(l-4)^2 + 2(k-5)^2$

$$\therefore F(l, k, g, \lambda) = 64g - 8l - 16k - \lambda(16g - 65 + 4(l-4)^2 + 2(k-5)^2)$$

$$F_l = -8 - 8\lambda(l-4) = 0 \quad (1)$$

$$F_k = -16 - 4\lambda(k-5) = 0 \quad (2)$$

$$F_g = 64 - 16\lambda = 0 \quad (3) \rightarrow \lambda = 4$$

$$F_\lambda = 0 \rightarrow 16g - 65 + 4(l-4)^2 + 2(k-5)^2 = 0 \quad (4)$$

$$(3) \text{ implies } \lambda = 4. \quad (1) \rightarrow l = \frac{15}{4} \quad (2) \rightarrow k = 4$$

$$(4) \rightarrow g = \frac{251}{64} \quad (\text{after substitution of } l \text{ & } k)$$

\therefore the CP of P (as a function of l, k and g)

$$\text{is } \left(\frac{15}{4}, 4, \frac{251}{64} \right).$$



(2) (a) Use LM to find the absolute extrema of (7)
 $f(x,y) = 2x - 3y + 5$ subject to $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(∴ it is implied in the question that absolute extrema do exist!)

Solution: Lagrangian is

$$F(x,y,\lambda) = 2x - 3y + 5 - \lambda \left(\frac{x^2}{9} + \frac{y^2}{16} - 1 \right)$$

$$F_x = 2 - \frac{2}{9}x\lambda = 0 \rightarrow x, \lambda \neq 0 \text{ and } x = \frac{9}{\lambda} \quad (1)$$

$$F_y = -3 - \frac{1}{8}y\lambda = 0 \rightarrow \lambda, y \neq 0 \text{ and } y = -\frac{24}{\lambda} \quad (2)$$

$$F_\lambda = 0 \rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1 \quad (3)$$

Substitute from (1)+(2) into (3). We get

$$\frac{\left(\frac{9}{\lambda}\right)^2}{9} + \frac{\left(-\frac{24}{\lambda}\right)^2}{16} = 1 \rightarrow \frac{9}{\lambda^2} + \frac{36}{\lambda^2} = 1 \therefore \lambda = \pm 3\sqrt{5}$$

$$\text{When } \lambda = 3\sqrt{5}, \quad x = \frac{3}{\sqrt{5}}, \quad y = -\frac{8}{\sqrt{5}}$$

$$f\left(\frac{3}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right) = \frac{30}{\sqrt{5}} + 5$$

$$\text{When } \lambda = -3\sqrt{5}, \quad x = -\frac{3}{\sqrt{5}}, \quad y = \frac{8}{\sqrt{5}}$$

$$f\left(-\frac{3}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right) = -\frac{30}{\sqrt{5}} + 5$$

∴ Absolute maximum of f subject to the constraint is $\frac{30}{\sqrt{5}} + 5$ at $\left(\frac{3}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$ and

the absolute minimum of f is $-\frac{30}{\sqrt{5}} + 5$ at $\left(-\frac{3}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$. ⑧

(b) The constraint curve has equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$

The general level curve for $f(x,y)$ is

$$f(x,y) = L = 2x - 3y + 5 \rightarrow y = \frac{2}{3}x + \frac{5-L}{3}$$

\therefore we find point(s) on the constraint curve where the tangent line has slope $\frac{2}{3}$

By implicit differentiation: $\frac{2}{9}x + \frac{1}{8}yy' = 0$

$\therefore y' = -\frac{16x}{9y}$ Equate to $\frac{2}{3}$ and manipulate:

$\frac{-16x}{9y} = \frac{2}{3} \rightarrow y = -\frac{8}{3}x$. Sub in constraint:

$$\frac{x^2}{9} + \frac{\left(\frac{-8}{3}x\right)^2}{16} = 1 \rightarrow \frac{16x^2 + 64x^2}{144} = 1$$

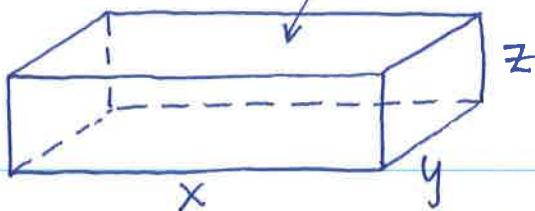
$$\rightarrow x^2 = \frac{144}{80} = \frac{9}{5} \quad \therefore x = \pm \frac{3}{\sqrt{5}}$$

When $x = \frac{3}{\sqrt{5}}$, $y = -\frac{8}{\sqrt{5}}$.

When $x = -\frac{3}{\sqrt{5}}$, $y = \frac{8}{\sqrt{5}}$.

\therefore the CP's of f satisfying the constraint are $\left(\frac{3}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$ and $\left(-\frac{3}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$ (as in (a)). ■

(3)



(9)

$$x = \text{length of box}, x > 0$$

$$y = \text{width " " }, y > 0$$

$$z = \text{height " " }, z > 0$$

We seek to maximize $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$

Lagrangian $F(x, y, z, \lambda) = V - \lambda g$ where

$$g = g(x, y, z) = 2xz + 2yz + xy - 12$$

$$\therefore F(x, y, z, \lambda) = xyz - \lambda(2xz + 2yz + xy - 12)$$

$$F_x = yz - \lambda(2z + y) = 0 \quad (1) \rightarrow yz = \lambda(2z + y)$$

$$F_y = xz - \lambda(2z + x) = 0 \quad (2) \rightarrow xz = \lambda(2z + x)$$

$$F_z = xy - \lambda(2x + 2y) = 0 \quad (3) \rightarrow xy = \lambda(2x + 2y)$$

$$F_\lambda = 0 \rightarrow 2xz + 2yz + xy = 12 \quad (4)$$

$$(5) = (1) \times x \rightarrow xyz = \lambda(2xz + xy) \quad (\because x, y, z \neq 0)$$

$$(6) = (2) \times y \rightarrow xyz = \lambda(2yz + xy) \quad \rightarrow xyz \neq 0, \quad \text{so } \lambda \neq 0 \text{ too}$$

$$(7) = (3) \times z \rightarrow xyz = \lambda(2xz + 2yz)$$

$$\therefore \text{by (5) + (6)} \quad \lambda 2xz = \lambda 2yz \quad \text{so } x = y$$

$$\text{From (7) + (5)} \quad \lambda xy = \lambda 2yz \quad \text{so } x = 2z$$

Substitution of $x = y = 2z$ in (4) gives

$$4z^2 + 4z^2 + 4z^2 = 12 \rightarrow z = 1 \quad (\because z > 0)$$

$\therefore x = y = 2$. Assuming that the box does

have a maximum volume subject to
the constraint (which is a very realistic
assumption), the maximum volume is

$$V(2,2,1) = 4 \text{ m}^3.$$



④ Here's a diagram of the fence enclosing the land:



$x = \text{length}, x > 0$
 $y = \text{width}, y > 0$

$$\text{Area is fixed at } A \rightarrow xy = A$$

(a) We use LM to minimize $c = c(x,y) = 2mx + 2ny$
subject to the constraint $xy = A$.

Lagrangian is $F(x,y,\lambda) = c(x,y) - \lambda g(x,y)$

$$\text{where } g(x,y) = xy - A$$

$$\therefore F(x,y,\lambda) = 2mx + 2ny - \lambda(xy - A)$$

$$F_x = 2m - \lambda y = 0 \quad ① \rightarrow \lambda, y \neq 0 \text{ and } y = \frac{2m}{\lambda}$$

$$F_y = 2n - \lambda x = 0 \quad ② \rightarrow x \neq 0 \text{ and } x = \frac{2n}{\lambda}$$

$$F_\lambda = 0 \rightarrow xy = A \quad ③ \quad \text{Substitute in } ③$$

$$\text{We get } \frac{2m}{\lambda} \cdot \frac{2n}{\lambda} = A \rightarrow \lambda^2 = \frac{4mn}{A}$$

$$\therefore \lambda \neq 0 \text{ (as } x, y > 0\text{)}, \lambda = \frac{2\sqrt{mn}}{\sqrt{A}}$$

$$\therefore x = \frac{2n\sqrt{A}}{2\sqrt{m}\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{m}}\sqrt{A}, y = \frac{2m\sqrt{A}}{2\sqrt{m}\sqrt{n}} = \frac{\sqrt{m}}{\sqrt{n}}\sqrt{A} \quad (11)$$

\therefore the plot of land that can be enclosed by a fence of least cost is $x = \frac{\sqrt{n}}{\sqrt{m}}\sqrt{A}$ long and $y = \frac{\sqrt{m}}{\sqrt{n}}\sqrt{A}$ wide.

Least cost of fence is

$$C\left(\frac{\sqrt{n}}{\sqrt{m}}\sqrt{A}, \frac{\sqrt{m}}{\sqrt{n}}\sqrt{A}\right) = \frac{2m\sqrt{n}\sqrt{A}}{\sqrt{m}} + \frac{2n\sqrt{m}\sqrt{A}}{\sqrt{n}}$$

$$= 4\sqrt{m}\sqrt{n}\sqrt{A}$$

(b) If $m=n$, then $x=\sqrt{A}=y$, so the plot is square!

(c) $y = \frac{A}{x}$ so we can write fence cost as a function of x only $C(x) = 2mx + \frac{2nA}{x}$

$$C'(x) = 2m - \frac{2nA}{x^2} = 0 \rightarrow x = \frac{\sqrt{n}\sqrt{A}}{\sqrt{m}}, y = \frac{\sqrt{m}\sqrt{A}}{\sqrt{n}}$$

$C''(x) = \frac{4nA}{x^3} > 0$, so C is least when x and y are as above

(d) There is no maximum fence cost. Here's why:

From $y = \frac{A}{x}$ let x be very large so $y \approx 0$

$\therefore C(x) = 2mx + 2n\left(\frac{A}{x}\right) \approx 2mx$ which is very large (and can be made larger)

than any given positive number). (12)

∴ $C(x)$ has no largest value so there is no maximum fence cost. ■

⑤ A production function is given by

$$P = Cx^a y^b \quad \text{where } x = \text{labour}, y = \text{capital}$$

and $C, a, b > 0$, and $a+b=1$

Unit labour cost is m and unit capital cost is n . Total budget is $T = mx + ny$ (so $T > 0$ is fixed)

Use LM to maximize P subject to the budget.

Solution : Constraint function is $g = mx + ny - T$

Lagrangian is $F(x, y, \lambda) = P - \lambda g$

$$= Cx^a y^b - \lambda(mx + ny - T)$$

$$F_x = cax^{a-1}y^b - m\lambda = 0 \rightarrow m\lambda = cax^{a-1}y^b \quad ①$$

$$F_y = cbx^a y^{b-1} - n\lambda = 0 \rightarrow n\lambda = cbx^a y^{b-1} \quad ②$$

$$F_\lambda = 0 \rightarrow mx + ny = T \quad (*)$$

$$③ = x \cdot ① : mx\lambda = cax^{a-1}y^b = aP \quad \left. \begin{array}{l} \therefore \lambda > 0 \text{ since} \\ \text{all other constants are} > 0 \end{array} \right\}$$

$$④ = y \cdot ② : ny\lambda = cbx^a y^{b-1} = bP$$

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$$\therefore \textcircled{3} \rightarrow P = \frac{mx\lambda}{a} \quad \text{and} \quad \textcircled{4} \rightarrow P = \frac{ny\lambda}{b}$$

$$\therefore \frac{mx}{a} = \frac{ny}{b} \rightarrow y = \frac{mbx}{na}$$

Substitute in (*) to get

$$mx + n\left(\frac{mbx}{na}\right) = T$$

$$mx\left(1 + \frac{b}{a}\right) = T \rightarrow x = \frac{aT}{m(a+b)} = \frac{aT}{m}$$

$$\therefore y = \frac{mb}{na} \left(\frac{aT}{m}\right) = \frac{bT}{n}$$

We conclude that P is maximized when

$$x = \frac{aT}{m} \quad \text{and} \quad y = \frac{bT}{n} \quad \text{and that the max}$$

$$\text{production is } P\left(\frac{aT}{m}, \frac{bT}{n}\right) = C \left(\frac{aT}{m}\right)^a \left(\frac{bT}{n}\right)^b \blacksquare$$

- ⑥ Assume $f(x, y, z) = x + 2y + 3z$ has absolute maximum & minimum subject to the two constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.

Use LM to find these absolute extrema.

Solution : Lagrangian is $F = f - \lambda g - \mu h$

where λ, μ are Lagrange multipliers,

$$g = g(x, y, z) = x - y + z - 1 \quad h = h(x, y, z) = x^2 + y^2 - 1$$

$$\therefore F(x, y, z, \lambda, \mu) = x + 2y + 3z - \lambda(x - y + z - 1) - \mu(x^2 + y^2 - 1)$$

We solve $F_x = F_y = F_z = F_\lambda = F_\mu = 0$.

$$F_x = 1 - \lambda - \mu 2x = 0 \quad ①$$

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$$F_y = 2 + \lambda - \mu 2y = 0 \quad ②$$

$$F_z = 3 - \lambda = 0 \quad ③ \rightarrow \lambda = 3$$

$$F_\lambda = 0 \rightarrow x - y + z = 1 \quad ④$$

$$F_\mu = 0 \rightarrow x^2 + y^2 = 1 \quad ⑤$$

Sub $\lambda = 3$ into ① to get $2\mu x = -2$

$$\text{so } \mu, x \neq 0 \text{ and } x = -\frac{1}{\mu} \quad ⑥$$

Sub $\lambda = 3$ into ② to get $5 = 2\mu y$

$$\text{so } \mu, y \neq 0 \text{ and } y = \frac{5}{2\mu} \quad ⑦$$

Use ⑥ ($x = -\frac{1}{\mu}$) and ⑦ ($y = \frac{5}{2\mu}$) and

$$\text{sub in ⑤ to get } \frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1 \therefore \mu^2 = \frac{29}{4}$$

$$\text{so } \mu = \pm \frac{\sqrt{29}}{2}.$$

$$\text{When } \mu = \frac{\sqrt{29}}{2}, \quad x = -\frac{2}{\sqrt{29}} \quad y = \frac{5}{\sqrt{29}} \quad z = 1 + \frac{7}{\sqrt{29}}$$

$$\text{When } \mu = -\frac{\sqrt{29}}{2}, \quad x = \frac{2}{\sqrt{29}} \quad y = -\frac{5}{\sqrt{29}} \quad z = 1 - \frac{7}{\sqrt{29}}$$

$$f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) = 3 + \sqrt{29} \rightarrow \text{Absolute MAX value}$$

$$f\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) = 3 - \sqrt{29} \rightarrow \text{Absolute MIN value}$$

(15)

- ⑦ (a) $y = h(x)$ is differentiable and $f(x,y) = C$ where C is a constant.

Write $z = f(x,y) = C$. By the chain rule we get $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\therefore \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

- (b) $y = h(x)$ satisfies the equation $f(x,y) = C$

The slope of the tangent line to the curve $f(x,y) = C$ at a point (x,y) on that curve is $y' = \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$

- (c) To apply LM to $f(x,y) = 2$ subject to

$g(x,y) = 0$, we solve $F_x = F_y = F_\lambda = 0$

where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

CP's are obtained as solutions to

$$F_x = 0 = f_x(x,y) - \lambda g_x(x,y) = 0 \quad ①$$

$$F_y = 0 = f_y(x,y) - \lambda g_y(x,y) = 0 \quad ②$$

$$F_\lambda = 0 \rightarrow g(x,y) = 0$$

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$$\textcircled{1} \rightarrow \lambda = \frac{f_x}{g_x} \quad \text{and} \quad \textcircled{2} \rightarrow \frac{f_y}{g_y} = \lambda$$

(assuming $g_x \neq 0$ and $g_y \neq 0$)

$\therefore \frac{f_x}{g_x} = \frac{f_y}{g_y}$ is a necessary condition for CP's (x, y) to satisfy.

The relevance of this to the method of LM is as follows. To optimize $f(x, y)$ subject to $g(x, y) = 0$ means to optimize C so that the level curve $f(x, y) = C$ intersects the constraint curve $g(x, y) = 0$. This happens when f & g touch (i.e. have a common tangent at a point (x, y)). This means

$$f_x = \lambda g_x \quad \text{and} \quad f_y = \lambda g_y$$

(i.e. $\frac{dy}{dx} = -\frac{f_x}{f_y}$ and $\frac{dy}{dx} = -\frac{g_x}{g_y}$ so

$$-\frac{f_x}{f_y} = -\frac{g_x}{g_y} \quad \text{so} \quad f_x = \lambda g_x \quad \text{and} \quad f_y = \lambda g_y$$

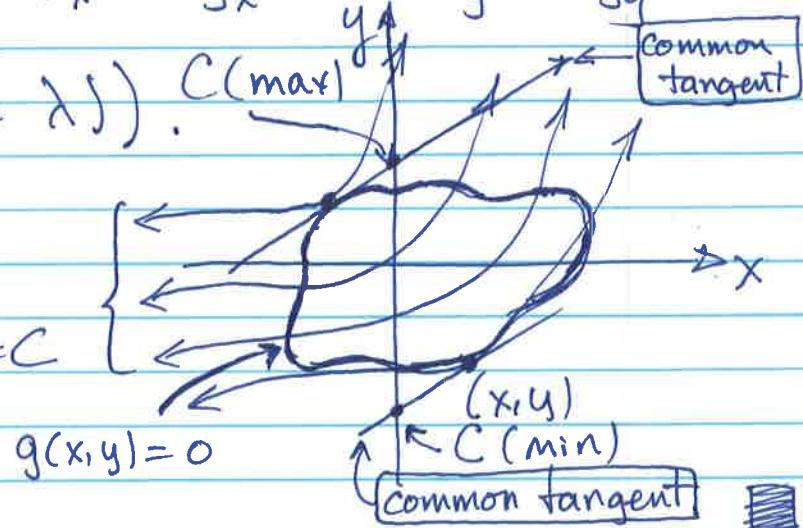
for some scalar (i.e. λ)).

(Draw a picture for problem ②)

Level curves

$$f(x, y) = C$$

Constraint.... $g(x, y) = 0$



⑧ Section 17.7 Problems

#10 We use LM to find the constrained critical point(s) of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints $x+y+z=4$ and $x-y+z=4$.

Solution Constraint functions are

$$g_1(x,y,z) = x+y+z-4 \text{ and } g_2(x,y,z) = x-y+z-4$$

Lagrangian is

$$F(x,y,z, \lambda, \alpha) = f(x,y,z) - \lambda g_1(x,y,z) - \alpha g_2(x,y,z)$$

λ, α are Lagrange multipliers.

We have 5 Lagrange equations :

$$F_x = 2x - \lambda - \alpha = 0 \quad \leftarrow ①$$

$$F_y = 2y - \lambda + \alpha = 0 \quad \leftarrow ②$$

$$F_z = 2z - \lambda - \alpha = 0 \quad \leftarrow ③$$

$$F_\lambda = -(x+y+z-4) = 0 \quad \leftarrow ④$$

$$F_\alpha = -(x-y+z-4) = 0 \quad \leftarrow ⑤$$

$$\begin{cases} ④ \Rightarrow x+y+z = 4 \\ ⑤ \Rightarrow x-y+z = 4 \end{cases} \quad \leftarrow \{ ④ - ⑤ \Rightarrow 2y = 0 \rightarrow y = 0$$

$$① - ③ \Rightarrow 2x - \lambda - \alpha = 2z - \lambda - \alpha$$

$$\therefore x = z \quad \text{Then } ④ \Rightarrow 2x = 4 \text{ so } x = z = 2$$

$$\therefore \text{the (only) CP is } (2, 0, 2)$$



(18)

#22 We have a utility function

$U(x, y) = 40x - 8x^2 + 2y - y^2$ where
 x, y are quantities of products X and Y.

Unit prices for X and Y respectively are

$P_x = 4$ and $P_y = 6$. The budget is

$I = 100$. We want to maximize U

subject to the constraint $4x + 6y = 100$

$$F(x, y, \lambda) = U(x, y) - \lambda(4x + 6y - 100)$$

$$= 40x - 8x^2 + 2y - y^2 - \lambda(4x + 6y - 100)$$

$$F_x = 40 - 16x - 4\lambda = 0 \quad \leftarrow \textcircled{1}$$

$$F_y = 2 - 2y - 6\lambda = 0 \quad \leftarrow \textcircled{2}$$

$$F_\lambda = -(4x + 6y - 100) = 0 \quad \leftarrow \textcircled{3}$$

$$\textcircled{1} \Rightarrow 10 - 4x - \lambda = 0 \Rightarrow -4x = \lambda - 10$$

$$\textcircled{2} \Rightarrow 6 - 6y - 18\lambda = 0 \Rightarrow -6y = 18\lambda - 6$$

$$\textcircled{3} \Rightarrow -4x - 6y + 100 = 0$$

$$\therefore \lambda - 10 + 18\lambda - 6 + 100 = 0$$

$$19\lambda + 84 = 0 \rightarrow \lambda = -\frac{84}{19}$$

$$\therefore x = -\frac{1}{4}\left(-\frac{84}{19} - 10\right) = \frac{137}{38} \quad y = -\frac{1}{6}\left(18\left(-\frac{84}{19}\right) - 6\right) = \frac{271}{19}$$

(19)

#24 Assume we have products X and Y whose unit prices are P_X and P_Y , respectively. Let $U(x, y)$ be a utility function of x and y, which are quantities of X and Y purchased, respectively. Assume a budget constraint $xP_X + yP_Y = I > 0$.

Show: If "satisfaction" (i.e. utility) is maximized at a point (x_0, y_0) , then

$$\lambda = \frac{U_x(x_0, y_0)}{P_X} = \frac{U_y(x_0, y_0)}{P_Y}$$

Proof: The LM theory tells that (x_0, y_0, λ) must be a constrained CP of the Lagrangian.

Lagrangian is $F = U - \lambda g$ where

$$g(x, y) = xP_X + yP_Y - I$$

$$\therefore F(x, y, \lambda) = U(x, y) - \lambda(xP_X + yP_Y - I)$$

Lagrange equations:

$$F_x = U_x(x, y) - \lambda P_X = 0 \quad ①$$

$$F_y = U_y(x, y) - \lambda P_Y = 0 \quad ②$$

$$F_\lambda = 0 \Rightarrow xP_X + yP_Y = I \quad ③$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \lambda = \frac{U_x(x_0, y_0)}{P_x} = \frac{U_y(x_0, y_0)}{P_y} \quad (20)$$

where (x_0, y_0, λ) satisfies $\textcircled{1}, \textcircled{2}$, and $\textcircled{3}$

That ends the proof. 

⑨ Let $u = f(x, y, z, w) = x^2 + 2y^2 + 3z^2 - w^2$

(a) Find and classify the extrema of f

Solution : $f_x = 2x = 0 \rightarrow x = 0$

$$f_y = 4y = 0 \rightarrow y = 0$$

$$f_z = 6z = 0 \rightarrow z = 0$$

$$f_w = -2w = 0 \rightarrow w = 0$$

\therefore the only CP of f is $(0, 0, 0, 0)$

Now we classify this CP

(that is what is meant by saying, "... classify the extrema of f ..." Even if the CP turns out not to yield extrema)

$$Hf(x, y, z, w) = \begin{pmatrix} \text{Matrix of} \\ \text{all 16 2nd} \\ \text{partials} \\ \text{of } f \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

(21)

We have $Hf(0,0,0,0) = Hf(x,y,z,w)$

and $\det(Hf(0,0,0,0)) = -96 < 0$

$\therefore f$ has no extrema at the CP $(0,0,0,0)$.

(b) The only CP of f is $(0,0,0,0)$ and $f(0,0,0,0) = 0$. We now show algebraically that f has no extrema at $(0,0,0,0)$

$f(x,0,0,0) = x^2 > 0$ for $x \neq 0$. Since x can be arbitrarily close to 0, but $x \neq 0$, we see that there are points close to $(0,0,0,0)$ ($\neq (0,0,0,0)$) for which $f > 0$.

$\therefore f$ cannot have a max @ $(0,0,0,0)$.

$f(0,0,0,w) = -w^2 < 0$ for $w \neq 0$. Since w can be arbitrarily close to 0, but $w \neq 0$, we see that there are points close to $(0,0,0,0)$ ($\neq (0,0,0,0)$) for which $f < 0$.

$\therefore f$ cannot have a min @ $(0,0,0,0)$.

We conclude that f has no extrema @ the CP $(0,0,0,0)$.



(10) Consider the function

$$f(x, y, z) = x^2 + y^2 + z^2 + (x^2 + y^2 + z^2)^{-1}$$

$$= x^2 + y^2 + z^2 + \frac{1}{x^2 + y^2 + z^2}$$

Observe that $f(x, y, z) = u(t)$

$$= t + \frac{1}{t} \quad (\text{where we let } t = x^2 + y^2 + z^2)$$

We look at $u(t)$. $u'(t) = 1 - \frac{1}{t^2}$

$$u'(t) = 0 \iff t = \pm 1$$

$$u''(t) = \frac{2}{t^3} > 0 \text{ if } t > 0.$$

For $t = x^2 + y^2 + z^2$, we can only use $t = 1$. And, u is minimized @ $t = 1$ (by 2nd DT from MATA32 above).

$\therefore f$ has a local minimum at all points (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$

Note also for such points where $t = x^2 + y^2 + z^2$ (so $t > 0$) we have that $u(t)$ ($x, y, z \in \mathbb{R}$) globally minimized. (ie the only critical point of u for $t > 0$ is at 1).

$\therefore f$ actually has a global minimum at all points (x, y, z) where $x^2 + y^2 + z^2 = 1$.



(11)

- (a) We need to maximize the profit function, $P(x, y, z) = 4x + 8y + 6z$, on the constraint $x^2 + 4y^2 + 2z^2 = 800$. To use Lagrange multipliers, we define a new function $h(x, y, z, \lambda) = 4x + 8y + 6z - \lambda(x^2 + 4y^2 + 2z^2 - 800)$. The constrained critical points of P are the critical points of h . Now $h_x = 4 - 2\lambda x$, $h_y = 8 - 8\lambda y$, $h_z = 6 - 4\lambda z$ and $h_\lambda = -(x^2 + 4y^2 + 2z^2 - 800)$. We put $h_x = h_y = h_z = h_\lambda = 0$. The first equation gives $x = \frac{2}{\lambda}$, the second gives $y = \frac{1}{\lambda}$ and the third gives $z = \frac{3}{2\lambda}$. The fourth equation now becomes $800 = \left(\frac{2}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{3}{2\lambda}\right)^2 = \frac{25}{2\lambda^2}$ so that $\lambda^2 = \frac{25}{1600}$ and $\lambda = \frac{1}{8}$ (we chose the positive root since we want a maximum). Hence the only suitable constrained critical point is $(16, 8, 12)$ and the corresponding profit is $4(16) + 8(8) + 6(12) = 200$ thousand dollars.

- (b) Now the constraint equation is $x^2 + 4y^2 + 2z^2 = 801$ and the first difference in solving occurs when solving for λ , where we now have $801 = \frac{25}{2\lambda^2}$. Hence, $\lambda \approx 0.12492$, $x \approx 16.009997$, $y \approx 8.004998$, $z \approx 12.007498$ and the maximum profit ≈ 200.12496 . Now the increase in profit is $\approx 200.12496 - 200 = 0.12496 \approx \lambda$. As you see from this calculation, the Lagrange multiplier λ is the instantaneous rate of change of the profit with respect to a change in the constraint.

(12)

- (a) To find the maximum production level we need to maximize $f(x, y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$ subject to the constraint $g(x, y) = 150x + 250y - 50,000 = 0$. Again we will use Lagrange multipliers. We put $h(x, y, \lambda) = 100x^{\frac{3}{4}}y^{\frac{1}{4}} - \lambda(150x + 250y - 50,000)$. Now $h_x = 75x^{-\frac{1}{4}}y^{\frac{1}{4}} - 150\lambda = 0$, $h_y = 25x^{\frac{3}{4}}y^{-\frac{3}{4}} - 250\lambda = 0$ and $h_\lambda = -(150x + 250y - 50,000) = 0$. Clearly $x \neq 0$ and $y \neq 0$ or else there is no production.

From the first equation we get $\lambda = \frac{x^{-\frac{1}{4}}y^{\frac{1}{4}}}{2}$. After substituting for λ in the second equation we get $x = 5y$. Now the third equation gives $150(5y) + 250y = 50,000 \implies y = 50 \implies x = 250$. Hence $f(250, 50) \approx 16718.5076$.

Since there were only finite resources, there must be a maximum production level, hence we have found it. The maximum production is 16,718 units.

- (b) From part (a) $\lambda = \frac{x^{-\frac{1}{4}}y^{\frac{1}{4}}}{2}$. When $x = 250$ and $y = 50$, the marginal productivity of money is $\lambda = \frac{(250)^{-\frac{1}{4}}(50)^{\frac{1}{4}}}{2} \approx 0.33437$. For each additional dollar spent on production, an additional 0.33437 unit of product can be produced.

END