University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT A33H **Winter 2011**

Solutions #6

Section 2.8 (12ed. 17.1)

2. $f(x, y) = 3x^2y - 4y$. $f(2, -1) = 3(2)^2(-1) - 4(-1) = -12 + 4 = -8$.

4. $g(x, y, z) = x^2yz + xy^2z + xyz^2$. $g(3, 1, -2) = (3)^2(1)(-2) + (3)(1)^2(-2) + (3)(1)(-2)^2 = -18 - 6 + 12 = -12$.

6. $h(r, s, t, u) = ru$. $h(1, 5, 3, 1) = (1)(1) = 1$.

6*. $h(r, s, t, u) = \ln(ru)$. $h(1, 5, 3, 1) = \ln\big((1)(1)\big) = \ln 1 = 0$.

8. $g(p_A, p_B) = p_A^2\sqrt{p_B} + 9$. $g(4, 9) = (16)(3) + 9 = 57$.

8*. $g(p_A, p_B) = p_A\sqrt{p_B} + 10$. $g(8, 4) = 8\sqrt{4} + 10 = 8(2) + 10 = 16 + 10 = 26$.

10. $F(x, y, z) = \dfrac{2x}{(y+1)z}$. $F(1, 0, 3) = \dfrac{2}{(1)(3)} = \dfrac{2}{3}$.

12. $f(x, y) = x^2y - 3y^3$. $f(r + t, r) = (r + t)^2r - 3r^3 = r^3 + 2r^2t + t^2r - 3r^3 = r(t^2 + 2rt - 2r^2)$.

14. The probability that, out of a total of four children, exactly three will be blue-eyed is

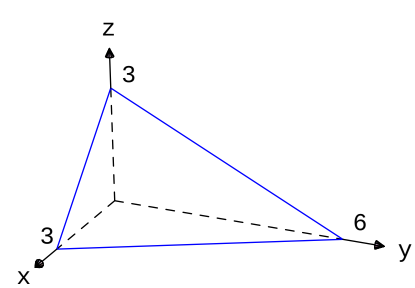
$$P(3, 4) = \frac{4\,!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{4-3}}{3\,!\,(4-3)\,!} = \frac{4\,!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)}{3\,!\,1\,!} = \frac{3}{64}\,.$$

16. A plane parallel to the yz–plane has the form $x = c$, where c is a constant. Since $(-2, 0, 0)$ is a point on this plane, the equation is $x = -2$.

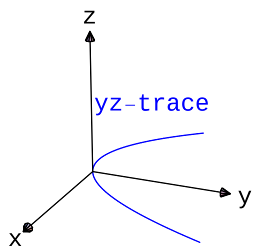
18. A plane parallel to the yz–plane has the form $x = c$, where c is a constant. Since $(96, -2, 2)$ is a point on this plane, the equation is $x = 96$.

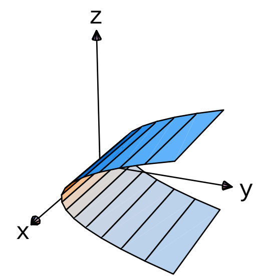
18*. A plane parallel to the yz–plane has the form $x = c$, where c is a constant. Since $(-4, -2, 7)$ is a point on this plane, the equation is $x = -4$.

20. The surface $2x + y + 2z = 6$ is a plane because its equation can be rewritten $2x + y + 2z - 6 = 0$, which is in the form of the equation of a plane ($Ax + By + Cz + D = 0$). The intercepts are $(3, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 3)$. The first quadrant part of the plane is illustrated on the right.

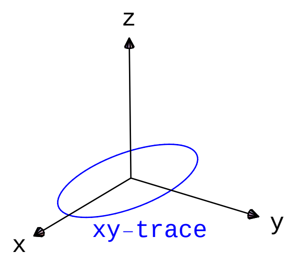


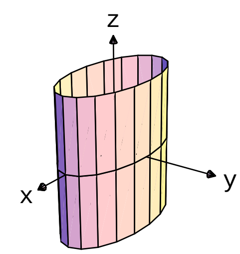
26. The surface $y = z^2$ has yz–trace $y = z^2$ and the section with any plane $x = a$ is also $y = z^2$. These are all parabolas. The surface is called a parabolic cylinder.



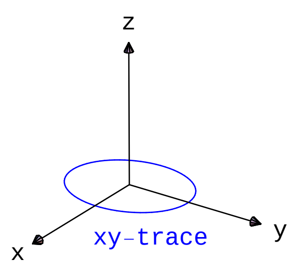


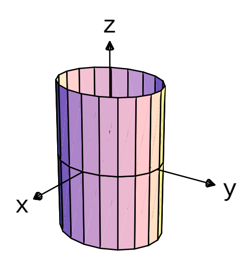
28. The surface $x^2 + 4y^2 = 1$ has xy–trace and section with any plane $z = c$ given by $x^2 + 4y^2 = 1$, which are ellipses. The surface is called an ellipitical cylinder.





28*. The surface $3x^2+2y^2 = 1$ has xy–trace and section with any plane $z = c$ given by $3x^2 + 2y^2 = 1$, which are ellipses. The surface is called an ellipitical cylinder.





Section 17.1 (12ed. 17.2)

2. $f(x, y) = 2x^2 + 3xy \qquad f_x(x, y) = 4x + 3y \qquad f_y(x, y) = 3x$

4. $f(x, y) = \ln 2 \qquad f_x(x, y) = 0 \qquad f_y(x, y) = 0$

6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$
$g_x(x, y) = 2(x^2 + 1)(2x) + 0 + 5(1)y^3 - 2(2x)y^2 = 4x(x^2 + 1) + 5y^3 - 4xy^2$
$g_y(x, y) = 0 + 3(y^3 - 3)^2(3y^2) + 5x(3y^2) - 2x^2(2y) = 9y^2(y^3 - 3)^2 + 15xy^2 - 4x^2y$

6*. $g(x, y) = (x + 1)^2 + (y - 3)^3 + 5xy^3 - 2 \qquad g_x(x, y) = 2(x + 1) + 5y^3 \qquad g_y(x, y) = 2(y - 3)^2 + 15xy^2$

8. $g(w, z) = \sqrt[3]{w^2 + z^2} = (w^2 + z^2)^{\frac{1}{3}} \qquad g_w(w, z) = \dfrac{1}{3}(w^2 + z^2)^{-\frac{2}{3}}(2w) = \dfrac{2\,w}{3(w^2 + z^2)^{\frac{2}{3}}}$
$g_z(w, z) = \dfrac{1}{3}(w^2 + z^2)^{-\frac{2}{3}}(2z) = \dfrac{2\,z}{3(w^2 + z^2)^{\frac{2}{3}}}$

10. $h(u, v) = \dfrac{8\,u\,v^2}{u^2 + v^2} \qquad h_u(u, v) = \dfrac{8v^2(u^2 + v^2) - 8uv^2(2u)}{(u^2 + v^2)^2} = \dfrac{8\,v^2\,(v^2 - u^2)}{(u^2 + v^2)^2}$
$h_v(u, v) = \dfrac{16uv(u^2 + v^2) - 8uv^2(2v)}{(u^2 + v^2)^2} = \dfrac{16\,u^3\,v}{(u^2 + v^2)^2}$

12. $Q(\ell,k) = 2\,\ell^{0.38}\,k^{1.79} - 3\,\ell^{1.03} + 2\,k^{0.13}$
$Q_\ell(\ell,k) = 2(0.38)\ell^{-0.62}\,k^{1.79} - 3(1.03)\ell^{0.03} = 0.76\,\ell^{-0.62}\,k^{1.79} - 3.09\ell^{0.03}$
$Q_k(\ell,k) = 2\ell^{0.38}(1.79)k^{0.79} + 2(0.13)k^{-0.87} = 3.58\,\ell^{0.38}\,k^{0.79} + 0.26\,k^{-0.87}$

16. $z = (x^3+y^3)\,e^{xy+3x+3y}$
$z_x = \dfrac{\partial z}{\partial x} = (x^3+y^3)\left(e^{xy+3x+3y}(y+3)\right) + e^{xy+3x+3y}(3x^2) = \left(3x^2 + (x^3+y^3)(y+3)\right)e^{xy+3x+3y}$
$z_y = \dfrac{\partial z}{\partial y} = (x^3+y^3)\left(e^{xy+3x+3y}(x+3)\right) + e^{xy+3x+3y}(3y^2) = \left(3y^2 + (x^3+y^3)(x+3)\right)e^{xy+3x+3y}$

16*. $z = (x^2+y^2)\,e^{2x+3y+1}$ $\quad z_x = \dfrac{\partial z}{\partial x} = 2x\,e^{2x+3y+1} + (x^2+y^2)\,e^{2x+3y+1}(2) = (2x^2+2y^2+2x)\,e^{2x+3y+1}$
$z_y = \dfrac{\partial z}{\partial y} = 2y\,e^{2x+3y+1} + (x^2+y^2)\,e^{2x+3y+1}(3) = (3x^2+3y^2+2y)\,e^{2x+3y+1}$

20. $f(r,s) = \sqrt{r\,s}\,e^{2+r} = (r\,s)^{\frac{1}{2}}\,e^{2+r}$ $\quad f_r(r,s) = (r\,s)^{\frac{1}{2}}\,e^{2+r} + \dfrac{1}{2}(r\,s)^{\frac{-1}{2}}(s)\,e^{2+r} = \left(\sqrt{r\,s} + \dfrac{s}{2\sqrt{r\,s}}\right)e^{2+r}$
$f_s(r,s) = \dfrac{1}{2}(r\,s)^{\frac{-1}{2}}(r)\,e^{2+r} = \dfrac{r\,e^{2+r}}{2\sqrt{r\,s}}$

22. $f(r,s) = (5r^2+3s^3)\,(2r-5s)$ $\quad f_r(r,s) = 2(5r^2+3s^3) + 10\,r\,(2r-5s)$
$f_s(r,s) = -5(5r^2+3s^3) + 9\,s^2\,(2r-5s)$

26. $g(r,s,t,u) = r\,s\,\ln(t)\,e^u$ $\quad g_r(r,s,t,u) = s\,\ln(t)\,e^u$ $\quad g_s(r,s,t,u) = r\,\ln(t)\,e^u$
$g_t(r,s,t,u) = \dfrac{r\,s\,e^u}{t}$ $\quad g_u(r,s,t,u) = r\,s\,\ln(t)\,e^u$

26*. $g(r,s,t,u) = r\,s\,\ln(2t+5u)$ $\quad g_r(r,s,t,u) = s\,\ln(2t+5u)$ $\quad g_s(r,s,t,u) = r\,\ln(2t+5u)$
$g_t(r,s,t,u) = \dfrac{2\,r\,s}{2t+5u}$ $\quad g_u(r,s,t,u) = \dfrac{5\,r\,s}{2t+5u}$

28. $z = \sqrt{2x^3+5xy+2y^2}$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{6x^2+5y}{2\sqrt{2x^3+5xy+2y^2}}$ $\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}} = \dfrac{5}{2\sqrt{2}}.$

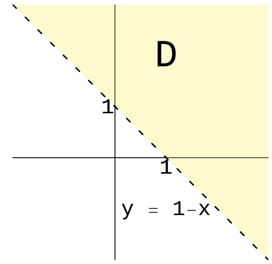
30. $g(x,y,z) = \dfrac{3x^2y^2+2xy+x-y}{xy-yz+xz}$
$g_y(x,y,z) = \dfrac{(xy-yz+xz)(6x^2y+2x-1)-(3x^2y^2+2xy+x-y)(x-z)}{(xy-yz+xz)^2}$ $\quad g_y(1,1,5) = 27.$

36. We are given $u = f(t,r,z) = \dfrac{(1+r)^{1-z}\,\ln(1+r)}{(1+r)^{1-z}-t}$.
Now $\dfrac{\partial u}{\partial z} = \ln(1+r)\,\left[\dfrac{(1+r)^{1-z}\,\ln(1+r)(-1)((1+r)^{1-z}-t) + (1+r)^{1-z}\,(1+r)^{1-z}\,\ln(1+r)}{((1+r)^{1-z}-t)^2}\right]$
$= \ln^2(1+r)\,(1+r)^{1-z}\left[\dfrac{-(1+r)^{1-z}+t+(1+r)^{1-z}}{((1+r)^{1-z}-t)^2}\right]$
$= \dfrac{t\,(1+r)^{1-z}\,\ln^2(1+r)}{((1+r)^{1-z}-t)^2}$.
This is what was required.

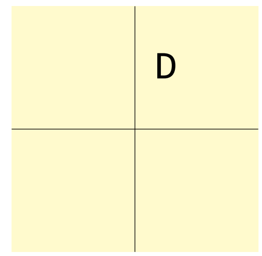
38. We are given that $r_L = r + D\,\dfrac{\partial r}{\partial D} + \dfrac{\partial C}{\partial D}$ and that elasticity is $\eta = \dfrac{\frac{r}{D}}{\frac{\partial r}{\partial D}}$. If we rewrite the formula for elasticity as $\dfrac{\partial r}{\partial D} = \dfrac{r}{D\,\eta}$ and substitute into the formula for r_L, we have $r_L = r + D\,\dfrac{r}{D\,\eta} + \dfrac{\partial C}{\partial D} = r\left(1+\dfrac{r}{\eta}\right) + \dfrac{\partial C}{\partial D} = r\,\left[\dfrac{1+\eta}{\eta}\right] + \dfrac{\partial C}{\partial D}$, as required.

 # 3.

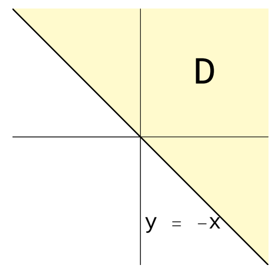
(a) Since $\ln t$ is only defined for $t > 0$, $f(x,y) = \ln(x+y-1)$ is only defined if $x+y-1 > 0$ or $y > 1-x$. Hence the domain is $D = \{(x,y) \in \mathbb{R}^2 \mid y > 1-x\}$. This is the region of the plane above the line $y = 1-x$, excluding the line.



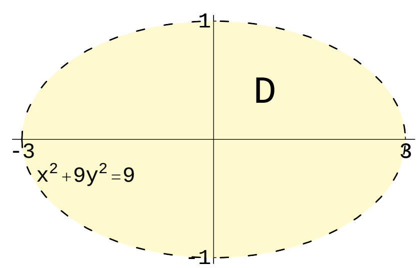
(b) $f(x,y) = e^{3\,x\,y}$ is defined for all pairs (x,y) of real numbers.



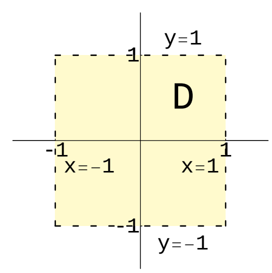
(c) Since $\sqrt{t}$ is only defined for $t \geq 0$, $f(x,y) = \sqrt{x+y}$ is defined for $y \geq -x$. Hence the domain is $D = \{(x,y) \in \mathbb{R}^2 \mid y \geq -x\}$. This is the region of the plane above the line $y = -x$, including the line.



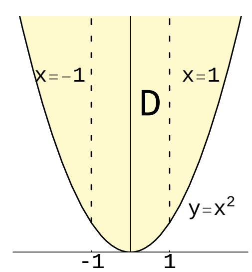
(d) Since $\ln t$ is only defined for $t > 0$, $f(x,y) = \ln(9 - x^2 - 9y^2)$ is only define if $9 - x^2 - 9y^2 > 0$ or $x^2 + 9y^2 < 9$. Hence the domain is $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 9y^2 < 9\}$. This is the region of the plane inside the ellipse $x^2 + 9y^2 = 9$, but excluding the ellipse.



(e) Since the domain of $\sqrt{1-x^2}$ is $-1 < x < 1$ and the domain of $\sqrt{1-y^2}$ is $-1 < y < 1$, the domain of $f(x,y) = \sqrt{1-x^2} - \sqrt{1-y^2}$ is $D = \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1, \ -1 < y < 1\}$. This is the part of the plane inside the square $[-1,\,1] \times [-1,\,1]$, but excluding the boundary lines.

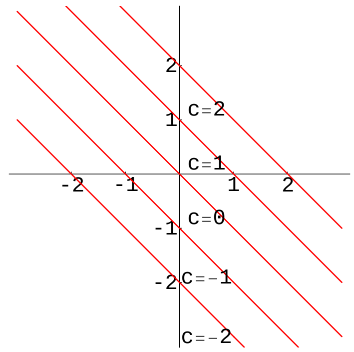


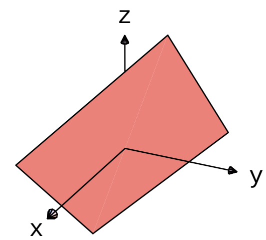
(f) We first note that $\sqrt{y-x^2}$ is only defined if $y - x^2 \geq 0$ or $y \geq x^2$ – the region of the plane above the graph of $y = x^2$, including the curve $y = x^2$. Also $\dfrac{1}{1-x^2}$ is only define if $1 - x^2 \neq 0$ or $x \neq \pm 1$. hence the domain of $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$ is $D = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2, \ x \neq \pm 1\}$. This is the region of the plane above the graph of $y = x^2$, including the curve $y = x^2$, but excluding the lines $x = -1$ and $x = 1$.



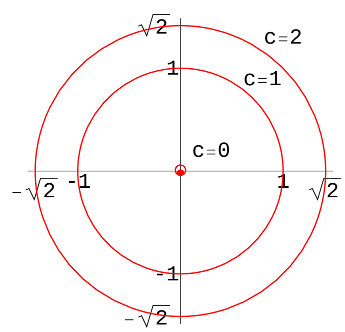
 # 4.

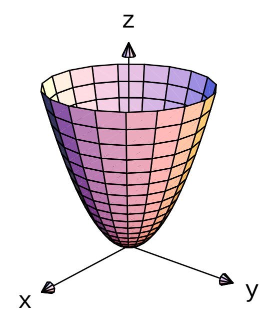
(a) The level curves are parallel lines. The graph of the function is a plane with xy–trace $y = -x$.



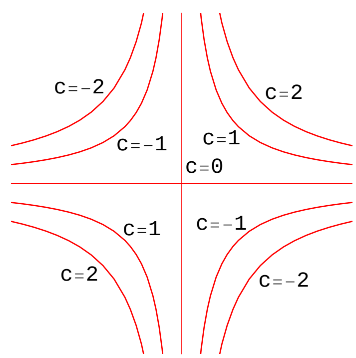


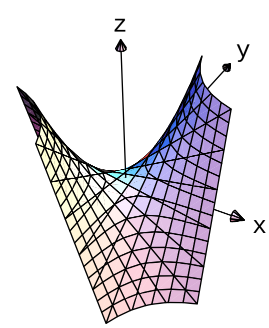
(b) The level curves are circles of radius $\sqrt{c}$ at the origin. Since $x^2 + y^2$ can never be negative, we can not draw level curves for $c = -2$ and $c = -1$. When $c = 0$, the level curve is only a point. The graph of the function is a paraboloid (bowl) opening upward with vertex at the origin.



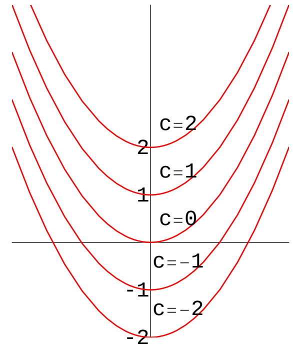


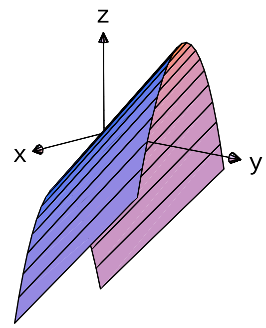
(c) The level curve for each c, $c \neq 0$ is a pair of hyperbolas in opposite quadrants of the plane. When $c = 0$, we get the x– axis and the y– axis. The graph of the function is saddle shaped.



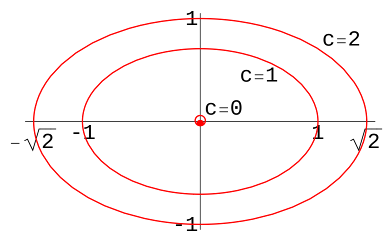


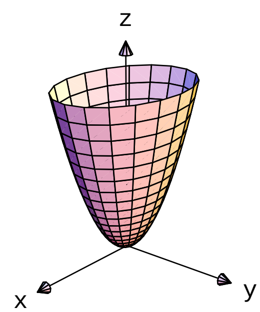
(d) The level curves are parabolas. The graph of the function has xy–trace the parabola $y = x^2$. Hence it is a parabolic cylinder.





(e) The level curves are ellipses centered at the origin. Since x^2+2y^2 can never be negative, we can not draw level curves for $c = -2$ and $c = -1$. When $c = 0$, the level curve is only a point. The graph of the function is an ellipitical paraboloid opening upward with vertex at the origin.





* question taken from the 12^{th} edition of the text.