SOLUTIONS

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test # 2

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: July 10, 2009
Duration: 110 minutes

Provide all of the following information:

(Print in Capitals) Lastname: SOLUTIONS

(Print) Given Name(s): ____________________

Student Number: ____________________

Signature: ____________________

Tutorial Number (e.g. TUT0033): ____________________

Circle the name of your Teaching Assistant:

Paula EHLERS   Xiangqun ZOU

Carefully read these instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure that all of these pages are included.
2. In Part A, enter your letter choice in the boxes at the top of page 2.
3. In Part B, put your solutions in the work space provided. If you need extra space, use the back of a page and clearly indicate the location of your continuing work.
4. You may use one standard hand-held calculator (graphing is acceptable). The following electronic devices are forbidden at your workspace: laptop computer, Blackberry, cell-phone, I-Pod, MP-3 player, or other similar electronic storage/retrieval devices.
5. Extra paper, notes (either visibly or in a pencil/carrying case), and textbooks are forbidden at your workspace.
6. Tests written in pencil will be denied any re-marking privilege.

SOLUTIONS

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5
b	e	a	d	e

Do not write anything in the boxes below.

Info	Part A
3	20

Part B

1	2	3	4	5
15	12	20	10	20

Total
100

Part A - Multiple Choice Questions Print the letter of the answer you think is correct in the mark box at the top of page 2. Each right answer earns 4 points. Each blank mark box or wrong answer earns 0 points. A small space is provided for your rough work.

1. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ -3 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix}$ then $det(BA + BC)$ equals

   (a) 114 (b) 18 (c) 66 (d) -18 (e) none of (a) - (d)

$\det(BA + BC)$

$= \det(B(A+C))$

$= \det(B)\det(A+C)$

$= (6)(3)$

$= 18$

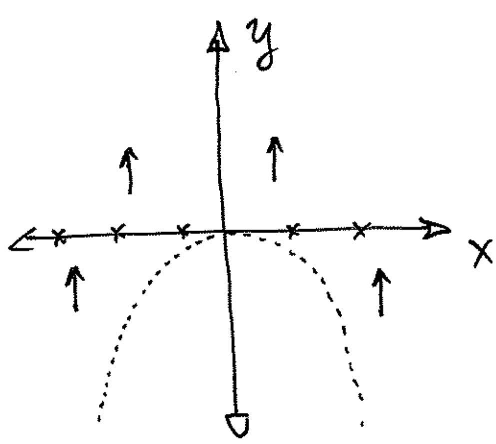
$A + C = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$

2. The domain of the function $f(x, y) = ln(y + x^2) + \frac{1}{y}$ is all points (x, y) that are

   (a) strictly above the x-axis
   (b) strictly above the curve $y = -x^2$
   (c) on or above the curve $y = -x^2$
   (d) on or above the curve $y = -x^2$ but not on the x-axis
   (e) strictly above the curve $y = -x^2$ but not on the x-axis
   (f) strictly above the curve $y = -x^2$ but not on the y-axis

We want $y + x^2 > 0$ and $y \neq 0$

$\therefore$ $y > -x^2$, $y \neq 0$



3. If $z = x^2 e^{3x+2y}$ then $z_x(1,-1)$ equals

(a) $5e$ (b) $5e^{-1}$ (c) $6e$ (d) $3e$ (e) none of (a) - (d)

$$z_x = 2x e^{3x+2y} + x^2 e^{3x+2y}(3)$$

$$z_x(1,-1) = 2e + 3e = 5e$$

4. The demand functions for products A and B are $a(x,y) = e^{-x-2y}$ and $b(x,y) = 5x^2y^{-1/2}$ where x and y are the unit prices of A and B respectively. We may conclude that the products A and B are

(a) complementary (b) competitive (c) both (a) and (b) (d) neither (a) nor (b)

$$\frac{\partial a}{\partial y} = e^{-x-2y}(-2) < 0 \quad \forall x,y > 0$$

$$\frac{\partial b}{\partial x} = 10xy^{-1/2} > 0 \quad \forall x,y > 0$$

5. Exactly how many of the following properties are equivalent to the statement:
"For a given natural number $n > 1$, the $n \times n$ matrix P is invertible" ?

✗ (i) $det(P) = 0$
✓ (ii) $PX = 0$ has only the trivial solution where 0 is the $n \times 1$ zero matrix
✓ (iii) $PY = B$ has a solution for every $n \times 1$ matrix B
✓ (iv) P^T is invertible
✓ (v) The reduced form of P is the $n \times n$ identity matrix

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5

Part B - Full Solution Problem Solving Full points are awarded for solutions that are numerically correct and sufficiently display concepts and methods in the curriculum of MATA33.

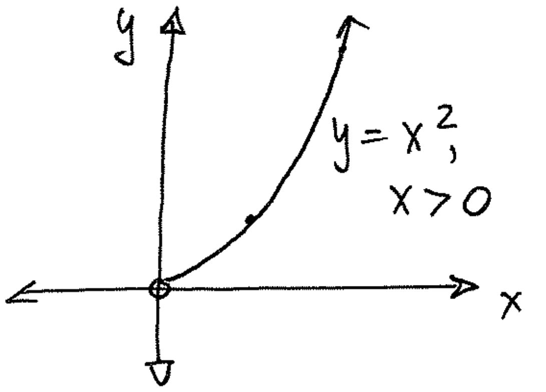
1. In all of this question let $H(x,y) = \dfrac{5x}{\sqrt{y}}$

(a) Find the function $y = f(x)$ that gives the level curve $L(5)$ of H. What is the domain of f that actually gives $L(5)$? Draw a small diagram of $L(5)$. [6 points]

$L(5) = \{ (x,y) \mid H(x,y) = 5 \}$

$\dfrac{5x}{\sqrt{y}} = 5 \iff x > 0$ and $x = \sqrt{y}$

$\therefore f(x) = x^2$ with domain $(0, \infty)$



Graph of $H(x,y) = 5$ is $L(5)$.

(b) Find $H_x(x,y)$ and $H_y(x,y)$. Express your answers using radicals, not exponents. [4 points]

$H_x(x,y) = \dfrac{5}{\sqrt{y}}$

$H_y(x,y) = 5x\left(-\frac{1}{2}\right) y^{-3/2}$

$= -\dfrac{5x}{2y\sqrt{y}}$

(c) Find the function $y = g(x)$ (and its domain) whose graph in the x, y-plane consists of the points (x,y) that satisfy the equation $H_x(x,y) = H_y(x,y)$ [5 points]

$\dfrac{5}{\sqrt{y}} = -\dfrac{5x}{2y\sqrt{y}} \iff x < 0,\ y > 0$ and $1 = -\dfrac{x}{2y}$

$\therefore y = -\dfrac{x}{2} = g(x)$ with domain $(-\infty, 0)$

2. Determine the value(s) of the real parameter s for which the linear system [12 points]

$$\left.\begin{aligned} 2sx + y &= 1 \\ 3sx + 6sy &= 2 \end{aligned}\right\} \circledast$$

has a unique solution and then explicitly find the solution (Hint: Cramer's rule)

$\circledast$ has a unique solution iff $\det(A) \neq 0$ where $A = \begin{pmatrix} 2s & 1 \\ 3s & 6s \end{pmatrix}$ is the coefficient matrix for $\circledast$.

$$\det(A) = 12s^2 - 3s = 3s(4s-1)$$

$\therefore$ $\det(A) \neq 0$ iff $s \in \mathbb{R}$, $s \neq 0, \frac{1}{4}$. These are the values of s for which $\circledast$ has a unique solution.

Now we find the unique solution. Let $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $s \in \mathbb{R}$, $s \neq 0, \frac{1}{4}$ we use Cramer's rule:

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 6s \end{vmatrix}}{\det(A)} = \frac{6s - 2}{3s(4s-1)} = \frac{2(3s-1)}{3s(4s-1)}$$

$$y = \frac{\begin{vmatrix} 2s & 1 \\ 3s & 2 \end{vmatrix}}{\det(A)} = \frac{4s - 3s}{3s(4s-1)} = \frac{s}{3s(4s-1)} = \frac{1}{3(4s-1)}$$

3. In all of this question let $A = \begin{pmatrix} -6 & -2 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(a) Use properties of determinants to find all real values of the variable x for which the matrix $F(x) = A - xI$ is invertible. [10 points]

$F(x)$ is invertible iff $\det(F(x)) \neq 0$.

$$F(x) = \begin{pmatrix} -6 & -2 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

$$= \begin{pmatrix} -6-x & -2 & 0 \\ 5 & 1-x & 0 \\ 0 & 0 & -x \end{pmatrix}$$

$$\det(F(x)) = -x\left[(-6-x)(1-x) + 10\right]$$

$$= -x\left[-6 + 6x - x + x^2 + 10\right]$$

$$= -x\left[x^2 + 5x + 4\right]$$

$$= -x(x+4)(x+1)$$

$\therefore$ $F(x)$ is invertible iff $x \neq -4, -1, 0$

Question 3 continued.

(b) Find the inverse of the matrix $F(-7)$. Write the entries in the inverse as rational numbers in lowest terms, not decimals. [10 points]

$$F(-7) = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 8 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$ We find $(F(-7))^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 5 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 18 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/7 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{18} & \frac{1}{18} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{7} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{9} & \frac{1}{9} & 0 \\ 0 & 1 & 0 & -\frac{5}{18} & \frac{1}{18} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{7} \end{array}\right)$$

$$\therefore (F(-7))^{-1} = \begin{pmatrix} \frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{5}{18} & \frac{1}{18} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}$$

4. Assume the equation $\overbrace{e^{xz} = xyz}^{(*)}$ defines the variable z implicitly as a function of the two independent variables x and y.

Evaluate z_y at the point (x, y, z) where $x = 1$ and $z = -1$. A complete answer requires that you find the value of y for the point (x, y, z). [10 points]

To find y when $x = 1$ and $z = -1$ sub in (*)

$$e^{-1} = -y \Rightarrow y = -\frac{1}{e}$$

$\therefore$ the point (x, y, z) is $\left(1, -\frac{1}{e}, -1\right)$

Now we get z_y:

$$\frac{\partial}{\partial y}\left(e^{xz}\right) = \frac{\partial}{\partial y}(xyz)$$

$$e^{xz} \cdot z_y = xz + xy\,z_y$$

$$z_y\left(e^{xz} - xy\right) = xz$$

$$\therefore \quad z_y = \frac{xz}{e^{xz} - xy}$$

$$z_y\left(1, -\frac{1}{e}, -1\right) = \frac{-1}{e^{-1} + e^{-1}} = \frac{-1}{\left(\frac{2}{e}\right)} = -\frac{e}{2}$$

5. In all of this question assume $c(x, y) = x\sqrt{y}\sqrt{x+y}$ is a joint cost function in dollars where $x, y > 0$ are the numbers of units of products X and Y respectively.

(a) Find the marginal cost functions. Express your answers using radicals, not exponents. [8 points]

$$C_x(x,y) = \sqrt{y}\sqrt{x+y} + \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

$$C_y(x,y) = \frac{x\sqrt{x+y}}{2\sqrt{y}} + \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

(b) State the mathematical approximation involving the cost function for units $x = 36$, $y = 64$, $y = 65$, and one of the marginal cost functions in part (a). Evaluate these functions and comment on the accuracy of this approximation. [6 points]

Desired approximation statement is:

$$c(36,65) - c(36,64) \approx C_y(36,64) \quad ☺$$

$$c(36,65) = 36\sqrt{65}\sqrt{101} \approx 2{,}916.8888$$

$$c(36,64) = 36(8)(10) = 2880$$

$$C_y(36,64) = \frac{36(10)}{16} + \frac{36(8)}{20}$$

$$= 22.5 + 14.4 = 36.9$$

$$c(36,65) - c(36,64) \approx 36.8888$$

The approximation in ☺ is extremely accurate. The difference between left + right sides in ☺ is about .0112

Question 5 continued.

(c) Verify that if $y > \frac{x}{2}$ then $c_x(x,y) > c_y(x,y)$ [6 points]

Assume $y > \frac{x}{2} > 0$ ($\because$ $x, y > 0$ as given)

$$C_x(x,y) - C_y(x,y)$$

$$= \sqrt{y}\sqrt{x+y} + \frac{x\sqrt{y}}{2\sqrt{x+y}} - \frac{x\sqrt{x+y}}{2\sqrt{y}} - \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

$$= \sqrt{y}\sqrt{x+y} - \frac{x\sqrt{x+y}}{2\sqrt{y}}$$

$$= \underbrace{\frac{\sqrt{x+y}}{\sqrt{y}}}_{>0}\underbrace{\left(y - \frac{x}{2}\right)}_{>0} > 0 \quad \left(\text{as } y > \frac{x}{2}\right)$$

$\therefore$ $C_x(x,y) > C_y(x,y)$ as required.