University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test # 1

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: June 6, 2009
Duration: 110 minutes

Provide all of the following information:

(Print in Capitals) Lastname: SOLUTIONS

(Print) Given Name(s): ____________________

Student Number: ____________________

Signature: ____________________

Tutorial Number (e.g. TUT0033): ____________________

Circle the name of your Teaching Assistant:

Paula EHLERS

Xiangqun ZOU

Carefully read these instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure that all of these pages are included.
2. In Part A, enter your letter choice in the boxes at the top of page 2.
3. In Part B, put your solutions in the work space provided. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
4. You may use one standard hand-held calculator (graphing is acceptable). The following electronic devices are forbidden at your workspace: laptop computer, Blackberry, cell-phone, I-Pod, MP-3 player, or other similar electronic storage/retrieval devices.
5. Extra paper, notes (either visibly or in a pencil/carrying case), and textbooks are forbidden at your workspace.
6. Tests written in pencil will be denied any re-marking privilege.

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5
d	d	a	b	c

Do not write anything in the boxes below.

Info	Part A
3	20

Part B

1	2	3	4	5	6
20	10	10	13	13	11

Total
100

Part A - Multiple Choice Questions Print the letter of the answer you think is correct in the mark box at the top of page 2. Each right answer earns 4 points. Each blank mark box or wrong answer earns 0 points. A small space is provided for your rough work.

1. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ -3 & 6 \end{bmatrix}$ then $AB + A^T$ equals

(a) $\begin{bmatrix} -9 & 29 \\ -11 & -33 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 7 \\ 7 & 35 \end{bmatrix}$ (c) $\begin{bmatrix} -9 & -29 \\ -11 & 33 \end{bmatrix}$ (d) $\begin{bmatrix} -9 & 29 \\ -11 & 33 \end{bmatrix}$

(e) none of (a) - (d)

$$AB + A^T = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 30 \\ -15 & 28 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 29 \\ -11 & 33 \end{bmatrix}$$

2. Let $Z = ax + by$ where x and y are variables and $a, b > 0$ are constants. If (x, y) satisfies the constraints: $x \le y$, $x - 3y \ge -6$, and $x + y \ge 2$ then

✓ (a) the maximum value of Z occurs amongst $(0, 2)$, $(3, 3)$, and $(1, 1)$.

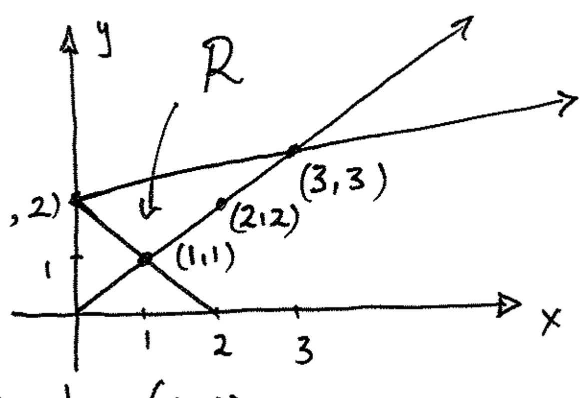
✓ (b) the minimum value of Z occurs amongst $(0, 2)$, $(3, 3)$, and $(1, 1)$.

(c) the minimum value of Z could also occur at the point $(2, 2)$.

(d) both (a) and (b) are true, but (c) is false.

(e) all of (a), (b), and (c) are true.

(c) is false. Here's why. If Z is minimized at $(2,2)$ then Z is minimized at all points joining $(1,1)$ to $(3,3)$



$\therefore$ minimum value of Z is $Z(1,1) = a+b$

so $ax+by = a+b$ so $y = -\frac{a}{b}x + \frac{a+b}{b}$.

This level curve has negative slope $-\frac{a}{b}$ which contradicts the slope of $y=x$ being 1.

3. How many 3×3 diagonal matrices D are there that satisfy $D^2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$?

(a) 4 (b) 3 (c) 2 (d) 1 (e) more than 4

$$D = \begin{pmatrix} \pm\sqrt{3} & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. Let $A = [a_{ij}]$ be the 9×9 upper triangular matrix where $a_{ij} = i^2 + j^2 - 4j$
The smallest element in A is

(a) -4 (b) -3 (c) -2 (d) 0 (e) none of (a) - (d)

$$a_{ij} = i^2 + j^2 - 4j$$
$$= i^2 + j^2 - 4j + 4 - 4$$
$$= i^2 + (j-2)^2 - 4 \geq -3 \quad \forall i,j$$

and $a_{12} = 1^2 + 0^2 - 4 = -3$

5. Exactly how many of the following statements are always true ?

✓ (i) Different 2×2 matrices can have the same reduced form.

✗ (ii) If the product of two matrices equals the zero matrix, then at least one of the two matrices must be the zero matrix.

✓ (iii) Every system of $m \geq 2$ homogeneous linear equations in $n > m$ unknowns has infinitely many solutions.

✗ (iv) A non-zero objective function defined on a non-empty feasible region has both a maximum and minimum value.

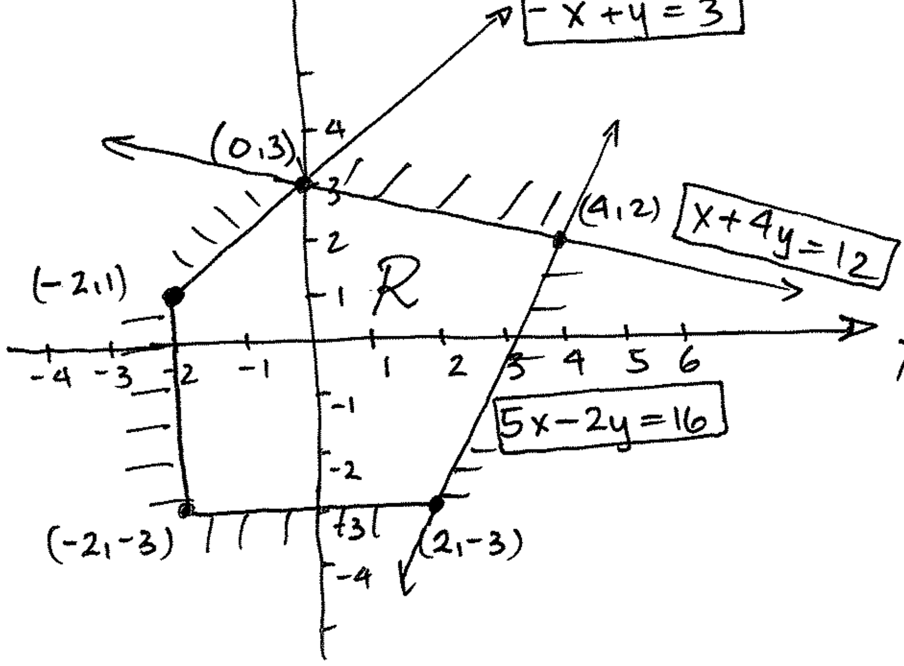
(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

Part B - Full Solution Problem Solving Full points are awarded for solutions that are numerically correct and sufficiently display concepts and methods in the curriculum of MATA33.

1. Find the maximum and minimum values of the function $Z = -2x - 8y$ subject to the five constraints: $x \geq -2$, $y \geq -3$, $-x + y \leq 3$, $x + 4y \leq 12$, and $5x - 2y \leq 16$.

   (To earn full points, your solution must include a neat, labeled diagram of the feasible region, the location of all points where Z is optimized, and all of your calculations.) [20 points]

R is non-empty & bounded.
Z is linear.
$\therefore$ Z is optimized on R by corner pt evaluation (by FTLP)



$-x+y=3 \rightarrow y = x+3 \quad (0,3), (-2,1)$

Test $(0,0)$: $0+0=0 \leq 3$ ✓

$x+4y=12 \quad (0,3), (4,2)$

Test $(0,0)$: $0+0=0 \leq 12$ ✓

$5x-2y=16 \quad (\frac{16}{5}, 0), (4,2), (2,-3)$

Test $(0,0)$: $0+0=0 \leq 16$ ✓

$Z(-2,-3) = 28$

$Z(-2,-1) = -4$

$Z(0,3) = -24$

$Z(4,2) = -24$

$Z(2,-3) = 20$

MAX $Z = 28$ @ $(-2,-3)$

MIN $Z = -24$ @ all points on segment joining $(0,3)$ to $(4,2)$

2. Use the method of reduction to solve the system of linear equations [10 points]

$$
\begin{aligned}
x + 3y - 4z &= -3 \\
5x - y + 2z &= 8 \\
-2x - 5y + 8z &= 8
\end{aligned}
$$

(To obtain full points, your answer must include the reduced form of the augmented matrix and must clearly state the solution to the system)

Augmented matrix :

$$\left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 5 & -1 & 2 & 8 \\ -2 & -5 & 8 & 8 \end{array}\right)$$

$$\begin{array}{l} R_2 - 5R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & -16 & 22 & 23 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$R_2 \leftrightarrow R_3 \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & -16 & 22 & 23 \end{array}\right)$$

$$R_3 + 16R_2 \to R_3 \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 22 & 55 \end{array}\right)$$

$$R_3\left(\tfrac{1}{22}\right) \to R_3 \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5/2 \end{array}\right)$$

$$R_1 + 4R_3 \to R_1 \left(\begin{array}{ccc|c} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5/2 \end{array}\right)$$

$$R_1 - 3R_2 \to R_1 \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5/2 \end{array}\right)}_{\text{Reduced matrix}}$$

Solution is :

$x = 1$

$y = 2$

$z = 5/2$

3. Let $M = \begin{bmatrix} -3 & -18 & -3 & 0 & -21 & 3 \\ 0 & 0 & 1 & 1 & 8 & 3 \\ 0 & 0 & 0 & -2 & -10 & -4 \end{bmatrix}$

State the system of linear equations whose augmented matrix is M. Find the reduced form of M and solve the system. [10 points]

Linear system is

$$\begin{aligned} -3x_1 - 18x_2 - 3x_3 \qquad\quad - 21x_5 &= 3 \\ x_3 + x_4 + 8x_5 &= 3 \\ -2x_4 - 10x_5 &= -4 \end{aligned}$$

$$M \to \left[\begin{array}{ccccc|c} 1 & 6 & 1 & 0 & 7 & -1 \\ 0 & 0 & 1 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \quad \left[\begin{array}{ccccc|c} 1 & 6 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 + R_3 \to R_1 \\ R_2 - R_3 \to R_2 \end{array} \quad \underbrace{\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right]}_{\text{Reduced form}}$$

Solution:

$x_1 = -2 - 6r - 4s$

$x_2 = r$ (free)

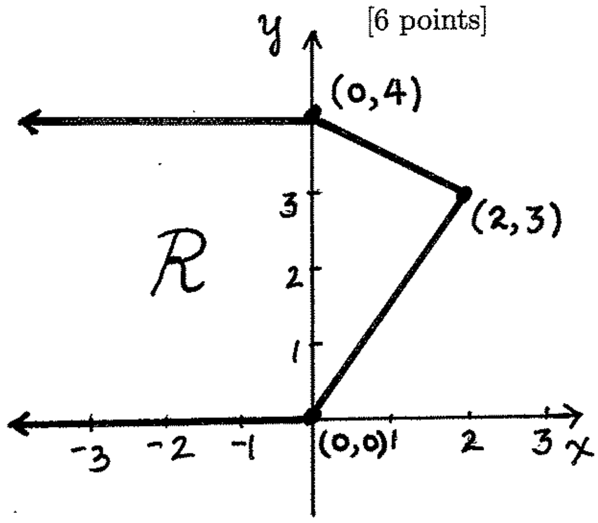
$x_3 = 1 - 3s$

$x_4 = 2 - 5s$

$x_5 = s$ (free)

4. Consider the unbounded region $\mathcal{R}$ illustrated below. The boundaries of R are included in R.

(a) State a system of four linear inequalities with integer coefficients whose solution is $\mathcal{R}$. [6 points]



System :

$y \geq 0$, $y \leq 4$

$-3x + 2y \geq 0$, $x + 2y \leq 8$

Details :

$y \geq \frac{3x}{2}$ for the pts $(0,0)$ & $(2,3)$

$\to -3x + 2y \geq 0$

$y = -\frac{x}{2} + 4$ for the points $(0,4)$ & $(2,3)$

(b) Let $0 < m < 1$. Show that every objective function of the form $Z = mx - y$ where $(x, y) \in \mathcal{R}$ has no minimum value but has a maximum value at $(0,0)$. [7 points]

No minimum : restrict Z to points (x,y) on the x-axis, $x \leq 0$.

$Z(x,0) = mx \to -\infty$ as $x \to -\infty$.

Maximum @ $(0,0)$: $Z(0,0) = 0$

If $Z = mx - y = k > 0$, then we have the level curve $y = mx - k$ which has $-k < 0$ as the y-intercept. For $0 < m < 1$, such a line does not touch R.

$\therefore$ if $Z = mx - y$ and $(x,y) \in R$, then $k \leq 0$ so Z is maximized at $(0,0)$, which is when $k = 0$.

5. A company makes and sells four products: P_1, P_2, P_3 and P_4. Each product has the same three types of production costs: c_1 is for materials, c_2 is for labour, and c_3 is for overhead. Consider the "product-cost" matrix

$$K = [k_{ij}] = \begin{bmatrix} 10 & 60 & 70 & 20 \\ 17 & 20 & 12 & 10 \\ 28 & 10 & 13 & 25 \end{bmatrix}$$

where k_{ij} = the number of cents per dollar of revenue from selling P_j that services cost c_i.

(For example, $k_{23} = 12$. This means 12 cents out of every dollar of revenue from selling product P_3 is committed to labour, which is cost c_2)

(a) State the matrices A and S such that the entries in AK are the average production costs per product and the entries in KS are the sum-total of each production cost. [2 + 2 points]

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Suppose there is a revision in the production costs as follows: all costs for P_1 increase by 6 cents; all costs for P_2 decrease by 5%; all costs for P_3 remain the same; all costs for P_4 increase by 10%. State the matrix C so that the sum $K + C$ has entries that reflect the revised production costs as described above. [5 points]

$$C = \begin{bmatrix} 6 & -3 & 0 & 2 \\ 6 & -1 & 0 & 1 \\ 6 & -.5 & 0 & 2.5 \end{bmatrix}$$

(c) Find a 4×4 diagonal matrix M such that the entries in KM give the revised production costs described in part (b), or explain why no such matrix M exists. [4 points]

Suppose M exists. Then M must look like

$$M = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & .95 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.1 \end{bmatrix}$$

We then need

$10a = 16$

$17a = 23$

$28a = 34$

But no $a \in \mathbb{R}$ satisfies all three conditions.

$\therefore$ no such matrix M exists.

6. (a) Assume that 2×2 matrices $P = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$ and $Q = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ commute. Find non-zero numbers a, b, and c such that $ax + by + cz = 0$ and $a > 0$. [6 points]

$$PQ = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} 2x+y & 2y+z \\ x-5y & y-5z \end{pmatrix}$$

$$QP = \begin{pmatrix} x & y \\ y & z \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 2x+y & x-5y \\ 2y+z & y-5z \end{pmatrix}$$

P & Q commute so $PQ = QP$

$\therefore\ 2y + z = x - 5y$

$\therefore\ x - 7y - z = 0$

Thus $a = 1$, $b = -7$, $c = -1$

(b) Let A be an $m \times n$ matrix ($m, n \geq 2$) and assume there is some $n \times m$ matrix C such that $CA = I$. Prove that the matrix equation $AX = B$ has a solution X for every $m \times 1$ matrix B. [5 points]

Let B be a general $m \times 1$ matrix.
Consider the equation $AX = B$
We get $C(AX) = CB$ so

$$(CA)X = CB$$

But $CA = I$ so $IX = CB$

$\therefore\ X = CB$, so we can solve the equation $AX = B$ for X given any B.