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MAT A33H **Winter 2011**

Reversing the Order of Integration

Let $z = f(x, y)$ and let B be a subset of the xy–plane. The volume of the figure lying below the graph of $f(x, y)$ and over the set B is denoted $\int_B f(x, y)\, dA$. To compute it requires a double integral. If B is the region between the curves $y = g(x)$ (below) and $y = h(x)$ (above) for x between a and b, then $\int_B f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$. Note that the limits of integration depend only the region B and are independent of the function $z = f(x, y)$ being integrated. Since the preceding discussion is symmetrical with respect to x and y, we can reverse the roles of x and y if convenient. That is, if B is written instead as the region bounded by the curves $x = p(y)$ (on the left) and $x = q(y)$ (on the right) for y between c and d then $\int_B f(x, y)\, dA$ is also given by $\int_B f(x, y)\, dA = \int_c^d \int_{p(y)}^{q(y)} f(x, y)\, dx\, dy$.

Example. Let $f(x, y) = xy + y^2$ and let B be the region between the curves $y = x$ and $y = x^2$. Compute $\int_B f(x, y)\, dA$.

Solution 1.

The curves $y = x$ and $y = x^2$ intersect at $x = 0$ and $x = 1$. For x between 0 and 1, $x^2 \leq x$. Therefore B is the region between $y = x^2$ (below) and $y = x$ (above) for x between 0 and 1. Therefore

$$\begin{aligned}
\int_B f(x, y)\, dA &= \int_0^1 \int_{x^2}^{x} (xy + y^2)\, dy\, dx = \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=x^2}^{y=x} dx \\
&= \int_0^1 \left(\frac{x^3}{2} + \frac{x^3}{3} - \frac{x^5}{2} - \frac{x^6}{3} \right) dx = \int_0^1 \left(\frac{5x^3}{6} - \frac{x^5}{2} - \frac{x^6}{3} \right) dx \\
&= \left[\frac{5x^4}{24} - \frac{x^6}{12} - \frac{x^7}{21} \right]_0^1 = \frac{5}{24} - \frac{1}{12} - \frac{1}{21} = \frac{13}{168}
\end{aligned}$$

Solution 2.

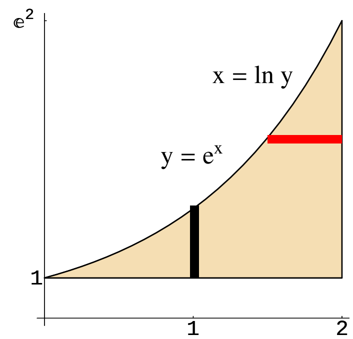
Solving for the curves for x in terms of y gives $x = y$ and $x = \sqrt{y}$ respectively. The curves $x = y$ and $x = \sqrt{y}$ intersect at $y = 0$ and $y = 1$. For y between 0 and 1, $y \leq \sqrt{y}$. Therefore

$$\begin{aligned}
\int_B f(x, y)\, dA &= \int_0^1 \int_{y}^{\sqrt{y}} (xy + y^2)\, dx\, dy = \int_0^1 \left[\frac{x^2 y}{2} + xy^2 \right]_{x=y}^{x=\sqrt{y}} dy \\
&= \int_0^1 \left(\frac{y^2}{2} + y^{5/2} - \frac{y^3}{2} - y^3 \right) dy = \int_0^1 \left(\frac{y^2}{2} + y^{5/2} - \frac{3}{2} y^3 \right) dy \\
&= \left[\frac{y^3}{6} - \frac{2y^{7/2}}{7} - \frac{3}{6} y^4 \right]_0^1 = \frac{1}{6} - \frac{2}{7} - \frac{3}{8} = \frac{13}{168}
\end{aligned}$$

as before. Note that the fact that the outer limits came out to be 0 and 1 after reversing the order, just as they were before, is a coincidence: the curves meet at the points $(0, 0)$ and $(1, 1)$ which happen to have the same x and y coordinates.

Example. Write $\int_0^2 \int_1^{e^x} f(x,y)\,dy\,dx$ with the order of integration reversed.

Solution. First sketch the region B described by the limits. B is bounded by $y = 1$ (below) and $y = e^x$ above, between $x = 0$ and 2, so it is the somewhat triangular-shaped region pictured below.



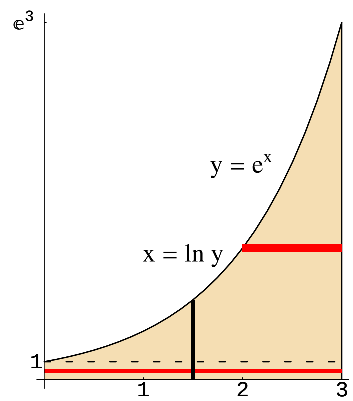
Solving $y = e^x$ for x gives $x = \ln y$. From the picture, we see that B can also be described as the region bounded by the curve $x = \ln y$ (on left) and $x = 2$ (on the right) for y between 1 and e^2. Therefore

$$\int_0^2 \int_1^{e^x} f(x,y)\,dy\,dx = \int_1^{e^2} \int_{\ln y}^{2} f(x,y)\,dx\,dy.$$

Sometimes reversing the order of integration, requires writing the region B as a union of regions.

Example. Write $\int_0^3 \int_0^{e^x} f(x,y)\,dy\,dx$ with the order of integration reversed.

Solution. Sketch the region B described by the limits:



We see that B is the union of the rectangle $[0,3] \times [0,1]$ and the somewhat triangular-shaped region above $y = 1$. Therefore the integral over B is the sum of two pieces so we get

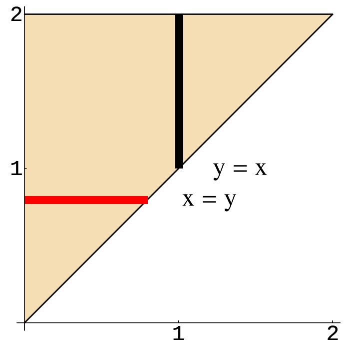
$$\int_0^3 \int_0^{e^x} f(x,y)\,dy\,dx = \int_0^1 \int_0^3 f(x,y)\,dx\,dy + \int_1^{e^3} \int_{\ln y}^{3} f(x,y)\,dx\,dy.$$

Now that we know that we **can** write an integral with the order of integration reversed, we can ask why we might **want** to reverse the order of integration. Consider the following example.

Example. Compute $\int_0^2 \int_x^2 e^{y^2}\, dy\, dx$

Solution. If we try to do the integral directly, we run into the problem that there is no elementary function whose derivative is e^{y^2}, so we cannot do the first integral. If we reverse the order of integration and integrate over x first, perhaps things will look different by the time we come to integrate over y.

The limits of integration describe the triangle bounded by $y = x$ (below) and $y = 2$ (above) for x between 0 and 2.



$$\int_0^2 \int_x^2 e^{y^2}\, dy\, dx = \int_0^2 \int_0^y e^{y^2}\, dx\, dy = \int_0^2 \left[x e^{y^2} \right]_{x=0}^{x=y} dy = \int_0^2 y e^{y^2}\, dy = \frac{1}{2} e^{y^2} \Big|_0^2 = \frac{1}{2}(e^4 - 1)$$