Vertical cross sections Let D be the region bounded below by $y = g_1(x)$, above by $y = g_2(x)$ and on the sides by $x = a$ and $x = b$. Then D can be described by

$$g_1(x) \leq y \leq g_2(x)$$

$$a \leq x \leq b$$

To integrate $f(x, y)$ over D, we have

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx\ .$$

Horizontal cross sections Let D be the region bounded on the left by $x = h_1(y)$, on the right by $x = h_2(y)$ and between the lines $y = c$ and $y = d$. Then D can be described by

$$h_1(y) \leq x \leq h_2(y)$$

$$c \leq y \leq d$$

To integrate $f(x, y)$ over D we have

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy\ .$$

Let $z = f(x, y)$ and let D be a subset of the xy–plane. The signed volume of the 3–dimensional region between D and the graph of f is called

$$\iint_D f(x, y)\, dA\ .$$

D can be almost any kind of a shape.

Volume as a double integral: If $f(x, y)$ is continuous and $f(x, y) \geq 0$ on the region D, then the solid region under the surface $z = f(x, y)$ over D has volume given by

$$V = \iint_D f(x, y)\, dA\ .$$

As a special case, when $f(x, y)$ is the constant function $f(x, y) = 1$, we have:

Area formula: The area of a region D in the xy–plane is given by the formula

$$\text{area of } D = A(D) = \iint_D 1\, dA\ .$$

Average Value formula: The average value of the function $f(x, y)$ over the region D is given by the formula

$$\text{Average Value} = \frac{1}{A(D)} \iint_D f(x, y)\, dA\ .$$

Example In a certain factory, output is given by the Cobb-Douglas production function

$$Q(K,\, L) = 50\, K^{\frac{3}{5}}\, L^{\frac{2}{5}}$$

where K is the capital investment in units of \$1,000 and L is the size of the labour force in worker-hours. Suppose that monthly capital investment varies between \$10,000 and \$12,000, while monthly use of labour varies between 2,800 and 3,200 worker-hours. Find the average monthly output for the factory.