The method of Lagrange multipliers is not restricted to a single constraint.

If $f(x, y, z)$ were subject to 2 constraints, $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, we use 2 lambdas, λ_1 and λ_2, and define the Lagrangian by

$$F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1 \, g_1(x, y, z) - \lambda_2 \, g_2(x, y, z)$$

If we solve $F_x = F_y = F_z = F_{\lambda_1} = F_{\lambda_2} = 0$, the critical points of F will give the (constrained) critical points of f.

A manufacturer has \$ 600,000 to spend on the production of a certain product. If x units of capital and y units of labour are allocated, the number of units produced is given by $p = p(x, y) = 120\, x^{4/5}\, y^{1/5}$ If labour costs \$ 5000 per unit and capital costs \$ 3000 per unit, how many units of labour and capital are needed to maximize production ? What is the maximum production ?

The total cost of resources is $3000\, x + 5000\, y$ is restricted by the resources available, \$ 600,000. Hence we want to maximize $p(x, y) = 120\, x^{4/5}\, y^{1/5}$ subject to the constraint $g(x, y) = 3000\, x + 5000\, y - 600,000 = 0$. To solve we will use Lagrange multipliers. Defining the Lagrangian, we have

$$F(x, y, \lambda) = p(x, y) - \lambda\, g(x, y)$$

$$= 120\, x^{4/5}\, y^{1/5} - \lambda\, (3000\, x + 5000\, y - 600,000)$$

Now $F_x = 120 \left(\frac{4}{5} \right) x^{-1/5}\, y^{1/5} - 3000\, \lambda = 96\, x^{-1/5}\, y^{1/5} - 3000\, \lambda = 0$

$$F_y = 120\left(\frac{1}{5}\right) x^{4/5}\, y^{-4/5} - 5000\,\lambda = 24\, x^{4/5}\, y^{-4/5} - 5000\,\lambda = 0$$

$$F_\lambda = -(3000\,x + 5000\,y - 600,000) = 0.$$

We note that the third can be rewritten as $3\,x + 5\,y - 600 = 0$.

Solving the first and second for λ we have

$$\lambda = \frac{96}{3000}\, x^{-1/5}\, y^{1/5} = 0.032\, x^{-1/5}\, y^{1/5}$$

and

$$\lambda = \frac{24}{5000}\, x^{4/5}\, y^{-4/5} = 0.0048\, x^{4/5}\, y^{-4/5}.$$

Equating and multiplying by $x^{1/5}\, y^{4/5}$ we get

$$0.032\, y = 0.0048\, x \implies y = 0.15\, x.$$

Then substituting into the third we have

$$3\,x + 5(0.15\,x) = 600 \text{ or } 3.75\,x = 600 \implies x = 160.$$

Hence $y = (0.15)\,160 = 24$.

There 160 units of capital and 24 units of labour needed for a maximum production of $p(160, 24) \approx 13,138$ units.

Suppose M is the maximum or minimum value of $f(x, y)$ subject to the constraint $g(x, y) = c$. The Lagrange multiplier λ is the rate of change of M with respect to c, $\lambda = \dfrac{d\,M}{d\,c}$.

Here $\lambda \approx$ change in M resulting from a 1–unit increase in c.

Note: A problem where production is maximized subject to a cost constraint is called a **fixed budget problem**. In the context of such a problem, the Lagrange multiplier λ is called the **marginal productivity of money**.

(In an utility problem, the λ is called the **marginal utility of money**.)

Now suppose the manufacturer in the previous problem is given an extra \$1000 to spend on resources; i.e. \$601,000. Estimate the effect on the maximum production level.

Recall that $\lambda = 0.032\, x^{-1/5}\, y^{1/5} = 0.032\,(160)^{-1/5}\,(24)^{1/5} \approx 0.0219 \implies$ an increase in production of approximately 0.0219 units for each \$1 increase in the constraint. Here an extra \$1000 was available so production would increase by approximately $(0.0219)\,(1000) = 21.9$ units giving a new production of approximately $13,138 + 21.9 = 13,159.9$ units.

If we simply redid the calculations, we would get $x = 160.27$, $y = 24.04$ and $p \approx 13,159.82$, which is essentially the same as the estimate.