We now detail how the second derivative test works in the case of 3 variables.

Let $w = f(x, y, z)$ have continuous second partial derivatives near a critical point (a, b, c).

The Hessian matrix is

$$A = H\,f(a, b, c) = \begin{bmatrix} f_{xx}(a, b, c) & f_{xy}(a, b, c) & f_{xz}(a, b, c) \\ f_{xy}(a, b, c) & f_{yy}(a, b, c) & f_{yz}(a, b, c) \\ f_{xz}(a, b, c) & f_{yz}(a, b, c) & f_{zz}(a, b, c) \end{bmatrix}.$$

We first compute $\det A = \det(H\,f)$. If $\det A \neq 0$, we can proceed.

If $\det A_1 = \det \left[f_{xx}(a, b, c) \right] > 0$

$$\det A_2 = \det \begin{bmatrix} f_{xx}(a, b, c) & f_{xy}(a, b, c) \\ f_{xy}(a, b, c) & f_{yy}(a, b, c) \end{bmatrix} > 0$$

$$\det A_3 = \det A > 0$$

we have the sequence $+\ +\ +$ which indicates a relative minimum.

$$\begin{aligned} &\text{If } \det A_1 = \det \left[f_{xx}(a,b,c) \right] < 0 \\ &\quad \det A_2 = \det \begin{bmatrix} f_{xx}(a,b,c) & f_{xy}(a,b,c) \\ f_{xy}(a,b,c) & f_{yy}(a,b,c) \end{bmatrix} > 0 \\ &\quad \det A_3 = \det A < 0 \end{aligned}$$

we have the sequence $\;-\;+\;-\;$ which indicates a relative maximum.

If we have any other sequence, the critical point is neither a minimum nor a maximum.

If $\det A = 0$, we have the degenerate case and we can not determine if the critical point is a maximum, a minimum or neither from the second derivative test.

Suppose we have a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$. We define a new function F of 4 variables by

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda\,(g(x, y, z))$$

If (a, b, c, λ_0) is a critical point of F, then (a, b, c) is a critical point of f subject to the constraint $g(x, y, z) = 0$.

The λ is called a **Lagrange multiplier** and the new function F we defined is called the **Lagrangian** or the **Lagrange function**.

This approach to solving for (constrained) critical points is called the **method of Lagrange multipliers**.

(Although stated for 3 variables, this approach can easily be adapted for 2, 4, 5 or more variables.)

Suppose we have a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$. We define a new function $\boldsymbol{F}$ of 4 variables by

$$\boldsymbol{F}(x, y, z, \lambda) = f(x, y, z) - \lambda\,(g(x, y, z))$$

If (a, b, c, λ_0) is a critical point of $\boldsymbol{F}$, then (a, b, c) is a critical point of f subject to the constraint $g(x, y, z) = 0$.

Find the constrained critical points of

$$f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \qquad .$$

We define the Lagrangian

$$\begin{aligned} \boldsymbol{F}(x, y, z, \lambda) &= f(x, y, z) - \lambda\, g(x, y, z) \\ &= (x - 3)^2 + (y - 1)^2 + (z + 1)^2 - \lambda\,(x^2 + y^2 + z^2 - 4) \end{aligned}$$

and calculate the partials and equate to 0:

$$\begin{aligned} F_x = &\quad 2(x-3) - 2\lambda x &= 0\ . \\ F_y = &\quad 2(y-1) - 2\lambda y &= 0 \\ F_z = &\quad 2(z+1) - 2\lambda z &= 0 \\ F_\lambda = & -(x^2+y^2+z^2-4) &= 0 \end{aligned}$$

We will solve the first 3 in terms of λ and then substitute into the 4^{th}.

From the first: $x - 3 = \lambda\, x \implies x(1-\lambda) = 3 \implies x = \dfrac{3}{1-\lambda}$

We note that $1-\lambda \neq 0$ because, if $\lambda = 1$, we would have $2x-6-2x = 0 \implies -6 = 0$ which is impossible.

From the second: $y - 1 = \lambda\, y \implies y(1-\lambda) = 1 \implies y = \dfrac{1}{1-\lambda}$

From the third: $z + 1 = \lambda\, z \implies z(1-\lambda) = -1 \implies z = \dfrac{-1}{1-\lambda}$.

Now the fourth becomes $\dfrac{3^2}{(1-\lambda)^2} + \dfrac{1^2}{(1-\lambda)^2} + \dfrac{(-1)^2}{(1-\lambda)^2} = 4 \implies \dfrac{11}{(1-\lambda)^2} = 4 \implies (1-\lambda)^2 = \dfrac{11}{4} \implies 1-\lambda = \pm\dfrac{\sqrt{11}}{2} \implies \lambda = 1 \pm \dfrac{\sqrt{11}}{2}$

Hence $x = \dfrac{3}{1-\left(1 \pm \frac{\sqrt{11}}{2}\right)} = \pm\dfrac{6}{\sqrt{11}}$

$$y = \frac{1}{1-\left(1 \pm \frac{\sqrt{11}}{2}\right)} = \pm\frac{2}{\sqrt{11}}$$

$$z = \frac{-1}{1-\left(1 \pm \frac{\sqrt{11}}{2}\right)} = \mp\frac{2}{\sqrt{11}}.$$

We have 2 (constrainted) critical points

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right) \text{ and } \left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right).$$