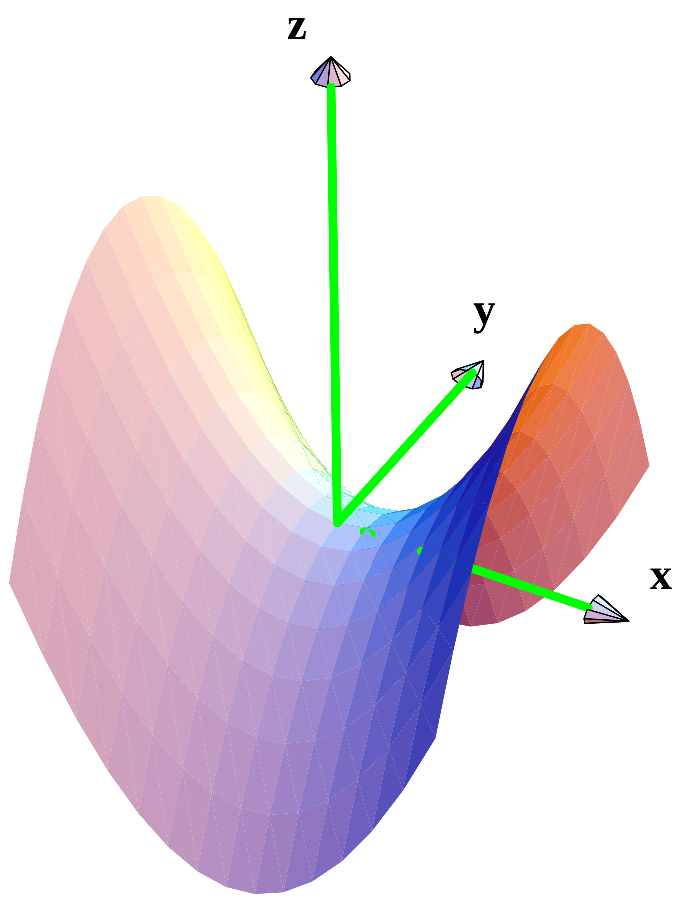
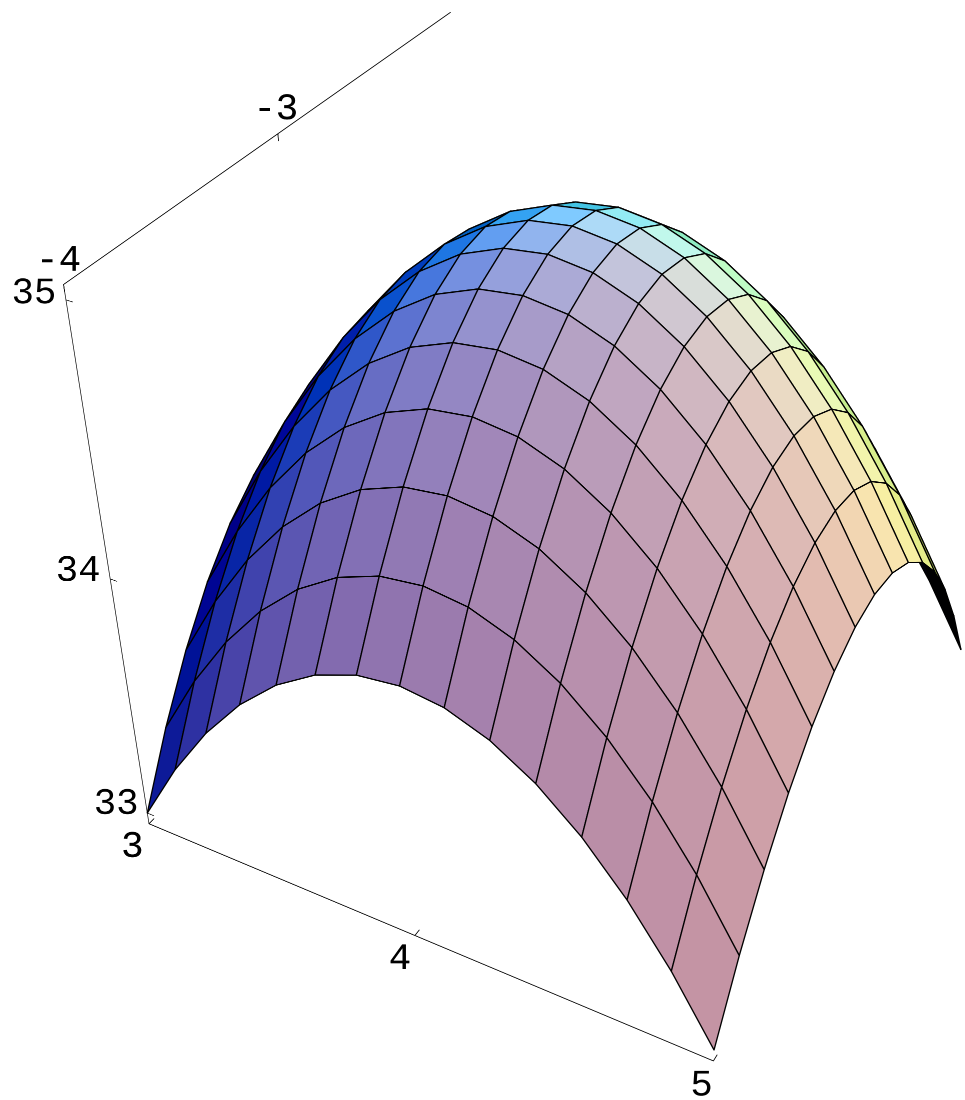
$$f(x, y) = x^2 - y^2$$

There is a critical point at $(0, 0)$, but no local extrema. Along the x–axis f increases away from $(0, 0)$, but along the y–axis f decreases away from $(0, 0)$.



$$f(x, y) = 10 + 8\,x - 6\,y - x^2 - y^2$$



Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{14} & a_{24} & a_{34} & a_{44} & \cdots & a_{4n} \\ & & \vdots & & & \\ a_{1n} & a_{2n} & a_{3n} & a_{4n} & \cdots & a_{nn} \end{bmatrix}$ be a symmetric matrix.

Define $A_k,\ k = 1, 2, \cdots, n$ to be the square matrix consisting of the first k rows and first k columns from A.

$$A_1 = [\, a_{11} \,], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}, \qquad \cdots$. (Called principal minors.)

Second Derivative Test   Suppose $z = f(x_1,\, x_2,\, \cdots,\, x_n)$ has continuous second partial derivatives at all points $(x_1,\, x_2,\, \cdots,\, x_n)$ near a critical point $\boldsymbol{a} = (a_1,\, a_2,\, \cdots,\, a_n)$. Let

$$A = Hf(\boldsymbol{a}) = \begin{bmatrix} f_{x_1x_1}(\boldsymbol{a}) & f_{x_1x_2}(\boldsymbol{a}) & \cdots & f_{x_1x_n}(\boldsymbol{a}) \\ f_{x_1x_2}(\boldsymbol{a}) & f_{x_2x_2}(\boldsymbol{a}) & \cdots & f_{x_2x_n}(\boldsymbol{a}) \\ & \cdots & & \\ f_{x_1x_n}(\boldsymbol{a}) & f_{x_2x_n}(\boldsymbol{a}) & \cdots & f_{x_nx_n}(\boldsymbol{a}) \end{bmatrix}$$

Case 1: $\det A \neq 0$.

1. If $\det A_1 > 0$, $\det A_2 > 0$, $\det A_3 > 0$, $\cdots$ (i.e., if $\det A_k > 0$ for $1 \leq k \leq n$) then $f(a_1,\, a_2,\, \cdots,\, a_n)$ is a relative (local) minimum.

2. If $\det A_1 < 0$, $\det A_2 > 0$, $\det A_3 < 0$, $\cdots$ (i.e., if $\det A_k$ has sign $(-1)^k$ for $1 \leq k \leq n$) then $f(a_1,\, a_2,\, \cdots,\, a_n)$ is a relative (local) maximum.

3. For any other sequence, $f(a_1,\ a_2,\ \cdots\ ,\ a_n)$ is neither a minimum nor a maximum.

Case 2: $\det A = 0$. This is the degenerate case. We can draw no conclusions — further analysis is required.

<u>Notes:</u>

1. Recall that $A_k,\ k = 1, 2, \cdots, n$ is the square matrix consisting of the first k rows and columns from A.

2. The matrix Hf is called the **Hessian matrix of** f and $Hf(\boldsymbol{a})$ is called the **Hessian matrix of** f **at** $\boldsymbol{a}$.