A Chain Rule : Let $z = f(x, y)$, where both x and y are functions of s and t given by $x = x(s, t)$ and $y = y(s, t)$. If f, x, and y all have continuous partial derivatives, then z is a function of s and t, and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The number of intermediate variables of z (two here) is the same as the number of terms that compose $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Another chain rule : Suppose $z = f(w, x, y)$ and w, x and y are all functions of p, q, r, s, and t. Then, as long as all partials are continuous, z can be considered a function of p, q, r, s, and t, and we have

$$\frac{\partial z}{\partial p} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial p} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p}$$
$$\frac{\partial z}{\partial q} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial q} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial q}$$
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The number of intermediate variables of z (three) is the same as the number of terms that form each of $\frac{\partial z}{\partial p}, \frac{\partial z}{\partial q}, \frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Another possibility : Suppose $w = f(x, y, z)$ is such that $x = x(t)$, $y = y(t)$ and $z = z(t)$. Then, as long as all derivatives are continuous, w can be considered a function of 1–variable, t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Since w can be expressed as a function of 1–variable, we use $\frac{dw}{dt}$ rather than $\frac{\partial w}{\partial t}$. Likewise we use $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$.

In general, if $z = f(x_1, x_2, \cdots, x_n)$ and each of $x_1, x_2, \cdots, x_n$ is a function of $u_1, u_2, \cdots, u_k$ we have

$$\frac{\partial z}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial u_j},$$

for $1 \leq j \leq k$.

Definition A function $z = f(x, y)$ is said to have a **relative** (or **local**) **maximum** at the point (a, b), if for all points (x, y) in the plane that are sufficiently close to (a, b), we have $f(a, b) \geq f(x, y)$.

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Definition A function $w = f(x, y, z)$ is said to have a **relative** (or **local**) **maximum** at the point (a, b, c), if for all points (x, y, z) in space that are sufficiently close to (a, b, c), we have $f(a, b, c) \geq f(x, y, z)$.

A function $w = f(x, y, z)$ is said to have a **relative** (or **local**) **minimum** at the point (a, b, c), if for all points (x, y, z) in space that are sufficiently close to (a, b, c), we have $f(a, b, c) \leq f(x, y, z)$.

A local (relative) maximum or minimum is called a **local** (or **relative**) **extremum**. (pl. extrema)

Rule 1: If $z = f(x, y)$ has a local (relative) maximum or minimum at (a, b), and if both f_x and f_y are defined for all points close to (a, b), it is necessary that (a, b) be a solution to the system

$$\begin{aligned} f_x(x, y) &= 0 \\ f_y(x, y) &= 0 \end{aligned} .$$

There is also a version of Rule 1 in the case of 3 or more variables.

Definition A point (a, b) for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$ is called a **critical point** of f.

Definition A point (a, b, c) for which $f_x(a, b, c) = 0$, $f_y(a, b, c) = 0$ and $f_z(a, b, c) = 0$ is called a **critical point** of f.

$$f(x,y) = \left(x^2 + y^2\right)^{\frac{2}{3}}$$