Let $A = \left[\, a_{ij} \,\right]$ be an $n \times n$ matrix. We define the **cofactor** of a_{ij} denoted c_{ij} by

$$c_{ij} = (-1)^{i+j} \det \tilde{A}_{ij} \ .$$

The $n \times n$ matrix $C = \left[\, c_{ij} \,\right]$ is called the **cofactor matrix** of A.

(Recall that $\tilde{A}_{ij}$ is the matrix obtained from A by deleting the i^{th} row and j^{th} column and is sometimes called the ij^{th} minor matrix of A. $\det \tilde{A}_{ij}$ can be called the ij^{th} **minor** of A or the **minor of the element** a_{ij} of A.)

In this context, a cofactor is sometimes called a signed minor.

<u>Note:</u> c_{ij} is a scalar (real number) but $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix.

We can now restate the definition of determinant in terms of cofactors.

(i) The determinant of a 1 by 1 matrix $\begin{bmatrix} a \end{bmatrix}$ is a.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} a_{1j}\, c_{1j} = a_{11}\, c_{11} + a_{12}\, c_{12} + \cdots + a_{1n}\, c_{1n}$$

where c_{ij} is the cofactor of a_{ij}.

The transpose of the cofactor matrix C of A is called the **classical adjoint** of A and is denoted by adj A ;i.e., adj $A \;=\; C^T$.

Theorem: If A is any square matrix, then

$$A\,(\,\mathrm{adj}\,A\,) = (\,\det A\,)\,I = (\,\mathrm{adj}\,A\,)\,A\,.$$

In particular, if $\det A \neq 0$, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A}\;\mathrm{adj}\,A$$

where adj $A \;=\; C^T$, C being the cofactor matrix of A.

We consider the linear system $A\,X = B$, where $A = \left[\,a_{i\,j}\,\right]$,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}.$$

If $\det A \neq 0$, we left multiply by A^{-1} to obtain

$$X = A^{-1}B = \left(\frac{1}{\det A}\ \mathrm{adj}\,A\right) B = \frac{1}{\det A} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ & \cdots & & \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

and we have

$$x_1 = \frac{1}{\det A}\left[B_1 c_{11} + B_2 c_{21} + \ \cdots\ + B_n c_{n1}\right]$$

$$x_2 = \frac{1}{\det A}\left[B_1 c_{12} + B_2 c_{22} + \ \cdots\ + B_n c_{n2}\right]$$

$$\cdots$$

$$x_n = \frac{1}{\det A}\left[B_1 c_{1n} + B_2 c_{2n} + \ \cdots\ + B_n c_{nn}\right].$$

If you look at the formula for x_1, it looks like the formula for expansion along the first column of the determinant of a matrix. The cofactors involved are those corresponding to the first column of A.

If A_1 is obtained from A by replacing the first column of A by B, then $c_{i\,1}(A_1) = c_{i\,1}(A)$ for each i. Therefore, expanding $\det A_1$ along the first column we have

$$\begin{aligned}
\det A_1 &= B_1 c_{11}(A_1) + \quad \cdots \quad + B_n c_{n1}(A_1) \\
&= B_1 c_{11}(A) + \quad \cdots \quad + B_n c_{n1}(A) \\
&= (\det A)\, x_1
\end{aligned}$$

$$\Longrightarrow \quad x_1 = \frac{\det A_1}{\det A}.$$

We get similar results for the other variables. This gives the theorem known as **Cramer's Rule**.

Cramer's Rule   If A is an invertible $n \times n$ matrix, the solution to the system

$$A\,X \;=\; B$$

of n equations in the variables $x_1,\ x_2,\ \cdots,\ x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing the k^{th} column of A by B.