Definition: determinant

(i) The determinant of a 1 by 1 matrix $\begin{bmatrix} a \end{bmatrix}$ is a.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

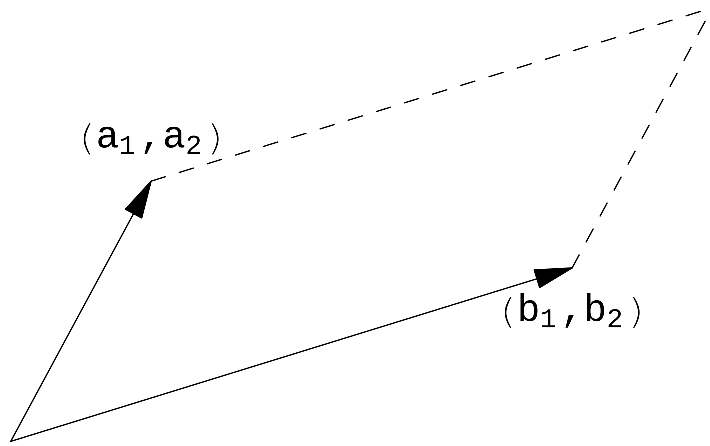
$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}$$

where $\tilde{A}_{ij}$ is the matrix obtained from A by deleting the i^{th} row and j^{th} column.

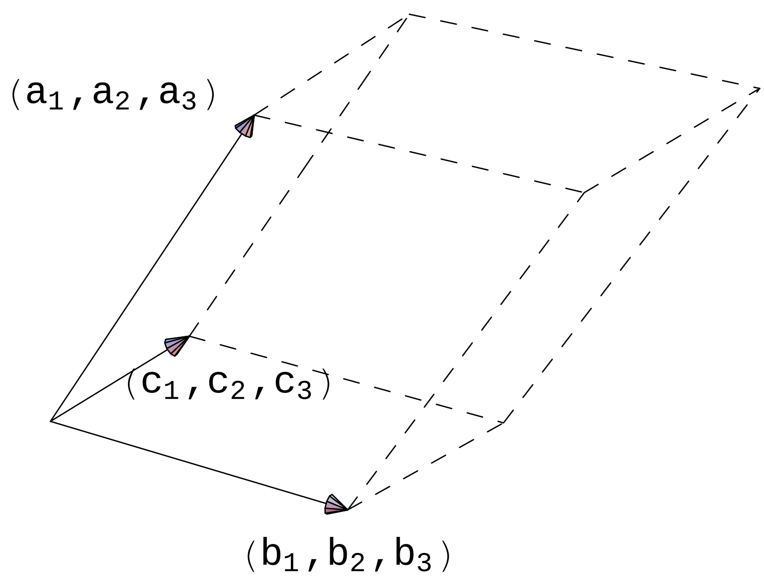
notes:

The matrix $\tilde{A}_{ij}$ is sometimes called the ij^{th} **minor matrix** of A.

Think of this approach as the definition by *expansion along the first row* or *expansion by minors along the first row.*



(a1,a2)
(b1,b2)



(a1,a2,a3)
(c1,c2,c3)
(b1,b2,b3)

Properties of Determinants

Let A be an $n \times n$ matrix.

- If A has a row or column of zeros, $\det A = 0$.

- If A is a triangular matrix, $\det A$ is the product of the elements on the main diagonal.

  - $\det I = 1$.

- $\det A \;=\; \det A^T$.

- Interchanging any two rows (or columns) multiplies the determinant by -1.

Since $\det A \;=\; \det A^T$, the remaining properties will be stated only in terms of rows.

- If a row of A is multiplied by a constant k, the determinant of the resulting matrix is $k \det A$.

  - NOTE, $\det(k\,A) = k^n \det A$.

- ▶ Let A, B, C be $n \times n$ matrices that are identical except that the i^{th} row of A is the sum of the i^{th} rows of B and C then $\det A = \det B + \det C$.
  - ▶ BUT, $\det A \neq \det B + \det C$ in general.
  - ▶ *This property + previous property* $\Longrightarrow$ *determinant is linear in each row.*
- ▶ If two distinct rows are identical then $\det A = 0$.
  - ▶ If two distinct rows are proportional then $\det A = 0$.
- ▶ Adding a multiple of one row to another row does not change the value of the determinant.
- ▶ $\det(A\,B) = (\det A)\,(\det B)$.

- ↯ A square matrix is invertible if and only if $\det A \neq 0$.

  - ► $$\det A^{-1} = \frac{1}{\det A}.$$

- ↯ The homogeneous linear system $A\,X = \mathbf{0}$ has the unique solution $X = \mathbf{0}$ if and only if $\det A \neq 0$.

- ↯ The determinant of a matrix consists of sums and products of its entries. If the entries are polynomials in some variable, say x, the the determinant is a polynomial in x. Often it is of interest to know the values of x that make the determinant zero.