Theorem: Let $A$ be a square matrix and let $A_k$ be the reduced matrix obtained from $A$ by a sequence $E_1, E_2, \cdots, E_k$ of elementary row operations. $A$ is invertible if and only if $A_k = I$.

Moreover, if $E_1, E_2. \cdots, E_k$ is a sequence of elementary row operations that takes $A$ to $I$, then the same sequence takes $I$ to $A^{-1}$.

Method to find the inverse of a matrix

If $A$ is an $n \times n$ matrix, form the $n \times (2n)$ matrix $\left[\, A \,\middle|\, I \,\right]$ and perform elementary row operations until the first $n$ columns form a reduced matrix. Assume the result is $\left[\, R \,\middle|\, B \,\right]$ so we have

$$\left[\, A \,\middle|\, I \,\right] \longrightarrow \cdots \longrightarrow \left[\, R \,\middle|\, B \,\right]$$

If $R = I$, then $A$ is invertible and $A^{-1} = B$.

If $R \neq I$, then $A$ is not invertible and $A^{-1}$ does not exist.

<u>Basic Properties of Matrix Inverse</u> Assume $A$, $B$ and the identity, $I$, are $n \times n$.

1. $I$ is invertible and $I^{-1} = I$.

2. If $A$ is invertible, so is $A^{-1}$ and $\left(A^{-1}\right)^{-1} = A$.

3. If $A$ and $B$ are invertible, so is $A\,B$ and

   $\left(A\,B\right)^{-1} = B^{-1}\,A^{-1}$.

4. If $A$ is invertible, so is its transpose $A^T$ and

   $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.