This is a collection of examples, prepared for recent lectures, but not used. In each case the topic(s) and lecture number is included.

Lecture 18 Chain Rule & implicit differentiation

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by

$$x^3 + y^3 + z^3 + 6xyz = 1 \ .$$

We first differentiate w.r.t. x giving

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy} \ .$$

Then differentiate w.r.t. y giving

$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy \frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy} \ .$$

We now work with a more general situation:

Suppose $F(x, y, z) = 0$ defines z implicitly as a function $z = g(x, y)$. Hence $F(x, y, g(x, y)) = 0$, all $(x, y) \in$ domain g. If F and g have continuous partial derivatives, we can use the chain rule to differentiate $F(x, y, z) = 0$ as follows

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \ .$$

But $\dfrac{\partial}{\partial x}(x) = 1$ and $\dfrac{\partial}{\partial x}(y) = 0$ so

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \ .$$

Hence

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z} \ .$$

Now differentiating w.r.t y we have

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \ .$$

So

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \ .$$

Hence

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z} \ .$$

We now return to the example where we were asked to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and put $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Now

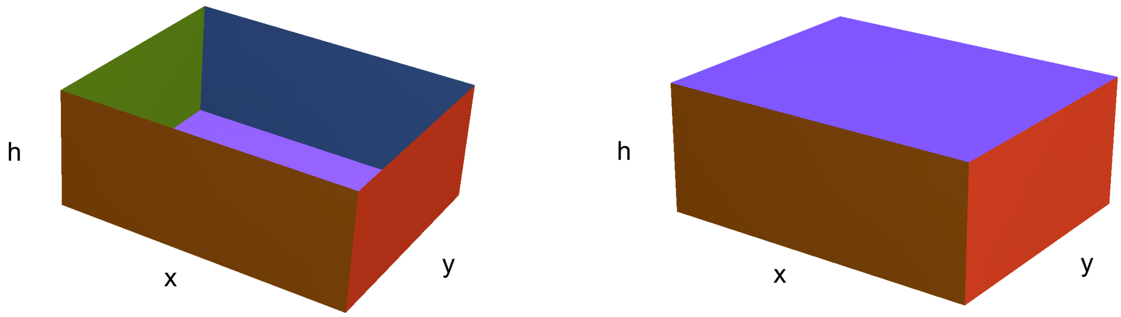
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\left(\frac{x^2 + 2yz}{z^2 + 2xy}\right).$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\left(\frac{y^2 + 2xz}{z^2 + 2xy}\right).$$

We note the results are the same as before.

Find the minimum cost of producing a rectangular box of volume $48\,m^2$. The front and back cost $\$1\,/m^2$, the top and bottom cost $\$2\,/m^2$ and the 2 sides cost $\$3\,/m^2$.



Let the length of the front be x, the length of a side be y and the height be h (as illustrated above). Since the volume $V = x\,y\,h = 48$ we have $h = \dfrac{48}{x\,y}$ (clearly, $x,\, y \neq 0$).

The cost function is
$$\begin{aligned} c(x,y) &= 2\,(2xy) + 1\,(2xh) + 3\,(2yh) \\ &= 4xy + 2x\left(\frac{48}{x\,y}\right) + 6y\left(\frac{48}{x\,y}\right) \\ &= 4xy + \frac{96}{y} + \frac{288}{x} \end{aligned}$$

Taking the partials and equating to 0 we have

$$c_x = 4y - \frac{288}{x^2} = 0 \implies y - \frac{72}{x^2} = 0 \implies xy = \frac{72}{x}$$
$$c_y = 4x - \frac{96}{y^2} = 0 \implies x - \frac{24}{y^2} = 0 \implies xy = \frac{24}{y}\,.$$

Hence we have $\dfrac{72}{x} = \dfrac{24}{y} \implies 72\,y = 24\,x \implies x = 3\,y$.

Now substituting into $x = \dfrac{24}{y^2}$ we have $3\,y - \dfrac{24}{y^2} = 0 \implies 3\,y^3 - 24 = 0 \implies y^3 = 8 \implies y = 2 \implies x = 6 \implies h = \dfrac{48}{(6)(2)} = 4$.

The Hessian matrix is $A = H\,f = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{xy} & c_{yy} \end{bmatrix} = \begin{bmatrix} \dfrac{576}{x^3} & 4 \\ 4 & \dfrac{192}{y^3} \end{bmatrix}$.

At the critical point, $(6,2)$, $A = \begin{bmatrix} \dfrac{8}{3} & 4 \\ 4 & 24 \end{bmatrix}$ so $\det A = 48 > 0$ and $\det A_1 = \dfrac{8}{3} > 0 \implies$ we have a relative minimum. Since there must be such a box, we have found it. The optimum box is $6\,m \times 2\,m \times 4\,m$ and costs $\$144.00$.

A retailer has discovered thet the number of TVs sold in a week is given by

$$\frac{5\,x}{2+x} + \frac{2\,y}{5+y}$$

where x is the expenditure on newspaper advertising and y is the expenditure on radio advertising. The profit per sale is \$ 250 and the total profit is

$$\Pi = \Pi(x, y) = 250\left(\frac{5\,x}{2+x} + \frac{2\,y}{5+y}\right) - x - y$$

Find the advertising expenditure that gives the maximum profit.

Now $\Pi_x = \dfrac{2500}{(2+x)^2} - 1 = 0$

$\Pi_y = \dfrac{2500}{(5+y)^2} - 1 = 0$.

The first gives $2500 = (2+x)^2 = 4 + 4x + x^2 \implies x^2 + 4x - 2496 = 0 \implies (x+52)(x-48) = 0$

The second gives $2500 = (5+y)^2 = 25 + 10y + y^2 \implies y^2 + 10y - 2475 = 0 \implies (y-45)(y+55) = 0$

Since $x,\ y \geq 0$, the only critical point is (48, 45).

The Hessian matrix is $A = Hf = \begin{bmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{xy} & \Pi_{yy} \end{bmatrix} = \begin{bmatrix} -\dfrac{5000}{(x+2)^3} & 0 \\ 0 & -\dfrac{5000}{(y+5)^3} \end{bmatrix}$.

At $(x, y) = (48,\, 45),\ A = \begin{bmatrix} -\dfrac{5000}{(48+2)^3} & 0 \\ 0 & -\dfrac{5000}{(45+5)^3} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{25} & 0 \\ 0 & -\dfrac{1}{25} \end{bmatrix}$.

Hence $\det A = \left(-\dfrac{1}{25}\right)\left(-\dfrac{1}{25}\right) = \dfrac{1}{625} \neq 0$ and $\det A_1 = -\dfrac{1}{25} < 0,\ \det A_2 = \det A = \dfrac{1}{625} > 0$. Therefore (48, 45) gives a relative maximum.

The maximal profit occurs when \$ 48 is spent on newspaper advertising and \$ 45 is spent on radio advertising.

A company has the possibility of discriminating between two markets where the demand functions are

$$\begin{aligned} q_1 &= 21 - \frac{1}{10}\,p_1 \\ q_2 &= 50 - \frac{2}{5}\,p_2 \end{aligned}$$

and the total cost function is

$$c = 2000 + 10\,q$$

$q = q_1 + q_2$. Find the amount to charge to maximize profit

(1) with price discrimination.

(2) without price discrimination, i.e., $p_1 = p_2$.

(1) Rewriting the demand functions we have

$$\begin{aligned} p_1 &= 210 - 10q_1 \\ p_2 &= 125 - \frac{5}{2}\,q_2 \end{aligned} .$$

The total cost is $c = 2000 + 10\,q_1 + 10\,q_2$ and the profit Π is

$$\begin{aligned} \Pi &= p_1\,q_1 + p_2\,q_2 - c \\ &= (210 - 10q_1)\,q_1 + \left(125 - \frac{5}{2}\,q_2\right)\,q_2 - (2000 + 10\,q_1 + 10\,q_2) \\ &= 200\,q_1 - 10\,q_1^2 + 115\,q_2 - \frac{5}{2}\,q_2^2 - 2000. \end{aligned}$$

Now $\Pi_{q_1} = 200 - 20\,q_1 = 0 \quad \Longrightarrow \; q_1 = 10$

$\Pi_{q_2} = 115 - 5\,q_2 = 0 \quad \Longrightarrow \; q_2 = 23$.

The Hessian matrix is $A = \begin{bmatrix} -20 & 0 \\ 0 & -5 \end{bmatrix}$ so $\det A_1 = -20 < 0$ and $\det A_2 = \det A = 100 > 0 \implies$ relative maximum.

The prices that give maximum profit are $p_1 = 210 - 10(10) = 110,\; p_2 = 125 - \frac{5}{2}\,(23) = 67.5$ and the maximum profit is

$$200(10) - 10(10)^2 + 115(23) - \frac{5}{2}\,(23)^2 - 2000 = 322.50\ .$$

(2) This time we need to use Lagrange.

Since $p_1 = p_2,\; 210 - 10\,q_1 = 125 - 2.5\,q_2 \implies 2.5q_2 - 10\,q_1 = -85$ which becomes our constraint. We now need to maximize profit subject to this constraint.

We define the Lagrangian,

$$F(q_1, q_2, \lambda) = 200\, q_1 - 10\, q_1^2 + 115\, q_2 - \frac{5}{2}\, q_2^2 - 2000 - \lambda\,(10\, q_1 - 2.5\, q_2 - 85)$$

Now $F_{q_1} = 200 - 20\, q_1 - 10\,\lambda = 0 \qquad \Longrightarrow\ q_1 = 10 - \dfrac{1}{2}\,\lambda$

$$F_{q_2} = 115 - 5\, q_2 + 2.5\,\lambda = 0 \qquad \Longrightarrow\ q_2 = 23 + \frac{1}{2}\,\lambda$$

$$F_\lambda = -(10\, q_1 - 2.5\, q_2 - 85) = 0\ .$$

Substituting for q_1 and q_2 in the third we have

$$10\left(10 - \frac{1}{2}\,\lambda\right) - 2.5\left(23 + \frac{1}{2}\,\lambda\right) - 85 = 0 \qquad \Longrightarrow\ \lambda = -6.8$$

Hence $q_1 = 10 - \dfrac{1}{2}(-6.8) = 13.4 \qquad \Longrightarrow\ p_1 = 210 - 10(13.4) = 76$

$$q_2 = 23 + \frac{1}{2}(-6.8) = 19.6 \qquad \Longrightarrow\ p_2 = 125 - \frac{5}{2}\,(19.6) = 76\ .$$

Now the maximum profit is

$$\Pi = (76)(13.4 + 19.6) - (2000 + 10(13.4) + 10(19.6) = 2508 - 2330 = 178\ .$$

If the company practices price discrimination the profit is \$ 322.50, while without price discrimination the profit is only \$ 178. It is more profitable to discriminate between the markets.

Find the (constrained) critical points of

$$f(x, y, z) = x\,z + y\,z$$

subject to $x^2 + z^2 = 2$ and $y\,z = 2$.

To solve, we put $g_1(x, y, z) = x^2 + z^2 - 2 = 0$ and $g_2(x, y, z) = y\,z - 2 = 0$ and define

$$F(x, y, z, \lambda_1, \lambda_2) = x\,z + y\,z - \lambda_1\,(x^2 + z^2 - 2) - \lambda_2\,(y\,z - 2)\ .$$

Now $F_x = z - 2\lambda_1\,x \quad = 0$

$F_y = z - \lambda_2\,z \quad = 0$

$F_z = x + y - 2\lambda_1\,z - \lambda_2\,y \quad = 0$

$F_{\lambda_1} = -(x^2 + z^2 - 2) \quad = 0$

$F_{\lambda_2} = -(y\,z - 2) \quad = 0$

From the second, $\lambda_2 = 1$ ($z = 0$ contradicts the fifth equation)

From the first, $\lambda_1 = \dfrac{z}{2\,x}$ ($x = 0 \implies z = 0$)

The third becomes $x + y - 2\left(\dfrac{z}{2\,x}\right) z - y = 0 \implies x^2 = z^2$

The fourth becomes $2\,x^2 = 2 \implies x^2 = 1 \implies x = \pm 1 \implies z = \pm 1$

Finally, the fifth gives $y = \dfrac{2}{z} = \pm 2$.

Hence we have 4 (constrained) critical points: $(1, 2, 1)$, $(1, -2, -1)$, $(-1, 2, 1)$ and $(-1, -2, -1)$.

You have to order supplies for an office. This week you need 3 products: A, B and C to be ordered in amounts x, y. and z, respectively. The total usefullness is modeled by

$$U(x, y, z) = xy + xyz \ .$$

If A costs \$ 3, B costs \$ 2, and C costs \$ 1 per unit and you have a budget of \$ 899, how much of each should you order?

We need to maximize $U(x, y, z) = xy + xyz$ subject to the constraint $3x + 2y + z = 899$. To do this we put

$$F(x, y, z, \lambda) = xy + xyz - \lambda(3x + 2y + z - 899)$$

Now $F_x = y + yz - 3\lambda = 0 \qquad \Longrightarrow \lambda = y\left(\frac{z+1}{3}\right)$

$F_y = x + xz - 2\lambda = 0 \qquad \Longrightarrow \lambda = x\left(\frac{z+1}{2}\right)$

$F_z = xy - \lambda = 0 \qquad \Longrightarrow \lambda = xy$

$F_\lambda = -(3x + 2y + z - 899) = 0$

Equating the second and third, we have $x\left(\frac{z+1}{2}\right) = xy \implies y = \frac{z+1}{2}$ $(x \neq 0 \text{ or } U = 0)$. Substituting into the first, we have $\lambda = \left(\frac{z+1}{2}\right)\left(\frac{z+1}{3}\right) = \frac{(z+1)^2}{6}$. Then substituting into the last, we have $3x + 2\left(\frac{z+1}{2}\right) + z = 899 \implies 3x = 898 - 2z \implies x = \frac{898-2z}{3}$. Now $xy = \lambda$ becomes $\left(\frac{898-2z}{3}\right)\left(\frac{z+1}{2}\right) = \frac{(z+1)^2}{6} \implies z = -1$ (not possible) or $\frac{z+1}{3} = \frac{898-2z}{3} \implies z + 1 = 898 - 2z \implies 3z = 897 \implies z = 299 \implies y = \frac{300}{2} = 150 \implies x = \frac{898-598}{3} = 100$.

Hence the most useful order is 100 of A, 150 of B and 299 of C.

As with many problems on this material, we are assuming that the critical points give the extrema.

A company has production function $f(x,y,z) = 50\,x^{2/5}\,y^{1/5}\,z^{1/5}$ which requires 3 imputs $x,\ y,\ z$, where x costs \$ 80 per unit, y costs \$ 12 per unit and z costs \$ 10 per unit. Find the number of units of each required to maximize production within a total budget of \$ 24,000. What is the maximum production?

We need to maximize $f(x,y,z) = 50\,x^{2/5}\,y^{1/5}\,z^{1/5}$ subject to the constraint $80\,x + 12\,y + 10\,z = 24000$. We will use Lagrange multipliers: Define

$$F(x,y,z,\lambda) = 50\,x^{2/5}\,y^{1/5}\,z^{1/5} - \lambda\,(80\,x + 12\,y + 10\,z - 24000)\ .$$

Now $F_x = 20\,x^{-3/5}\,y^{1/5}\,z^{1/5} - 80\,\lambda = 0 \qquad \Longrightarrow\ \lambda = \frac{1}{4}\,x^{-3/5}\,y^{1/5}\,z^{1/5}$

$F_y = 10\,x^{2/5}\,y^{-4/5}\,z^{1/5} - 12\,\lambda = 0 \qquad \Longrightarrow\ \lambda = \frac{5}{6}\,x^{2/5}\,y^{-4/5}\,z^{1/5}$

$F_z = 10\,x^{2/5}\,y^{1/5}\,z^{-4/5} - 10\,\lambda = 0 \qquad \Longrightarrow\ \lambda = x^{2/5}\,y^{1/5}\,z^{-4/5}$

$F_\lambda = -(80\,x + 12\,y + 10\,z - 24000) = 0\ .$

Eliminating z from the first two, we have $\frac{1}{4}\,x^{-3/5}\,y^{1/5} = \frac{5}{6}\,x^{2/5}\,y^{-4/5} \Longrightarrow x = \frac{3}{10}\,y\ .$

Eliminating x from the second and third, we have $\frac{5}{6}\,y^{-4/5}\,z^{1/5} = y^{1/5}\,z^{-4/5} \Longrightarrow z = \frac{6}{5}\,y\ .$

Substituting into the fourth, we have

$$24000 = 80\left(\frac{3}{10}\,y\right) + 12\,y + 10\left(\frac{6}{5}\,y\right) = 24\,y + 12\,y + 12\,y = 48\,y \Longrightarrow y = 500\ .$$

Hence $x = \frac{3}{10}\,(500) = 150$ and $z = \frac{6}{5}\,(500) = 600$.

Since we are assuming that extrema are attainable at critical points, the production is maximized within budget by using 150 units of x, 500 units of y and 600 units of z. The maximum production is $f(150,500,600) =$ 4622 units.