Definition: determinant

(i) The determinant of a 1 by 1 matrix $\begin{bmatrix} a \end{bmatrix}$ is a.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}$$

where $\tilde{A}_{ij}$ is the matrix obtained from A by deleting the i^{th} row and j^{th} column.

notes:

The matrix $\tilde{A}_{ij}$ is sometimes called the ij^{th} **minor matrix** of A.

Think of this approach as the definition by *expansion along the first row* or *expansion by minors along the first row.*

Properties of Determinants

Let A be an $n \times n$ matrix.

- If A has a row or column of zeros, $\det A = 0$.
- If A is a triangular matrix, $\det A$ is the product of the elements on the main diagonal.
  - $\det I = 1$.
- $\det A \ = \ \det A^T$.
- Interchanging any two rows (or columns) multiplies the determinant by -1.

Since $\det A \ = \ \det A^T$, I will state the remaining properties only in terms of rows.

♠ ► If a row of A is multiplied by a constant k, the determinant of the resulting matrix is $k \det A$.

► NOTE, $\det(k\,A) = k^n \det A$.

♠ ► Let A, B, C be $n \times n$ matrices that are identical except that the i^{th} row of A is the sum of the i^{th} rows of B and C then $\det A = \det B + \det C$.

► BUT, $\det A \neq \det B + \det C$ in general.

► *This property + previous property* $\Longrightarrow$ *determinant is linear in each row.*

♠ ► If two distinct rows are identical then $\det A = 0$.

► If two distinct rows are proportional then $\det A = 0$.

♠ ► Adding a multiple of one row to another row does not change the value of the determinant.

► $\det(A\,B) = (\det A)\,(\det B)$.

ϟ The determinant of a matrix consists of sums and products of its entries. If the entries are polynomials in some variable, say x, the the determinant is a polynomial in x. Often it is of interest to know the values of x that make the determinant zero.

ϟ A square matrix is invertible if and only if $\det A \neq 0$.

ϟ The homogeneous linear system $A\,X = \mathbf{0}$ has the unique solution $X = \mathbf{0}$ if and only if $\det A \neq 0$.

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ matrix. We define the **cofactor** of a_{ij} denoted c_{ij} by

$$c_{ij} = (-1)^{i+j} \det \tilde{A}_{ij} \ .$$

The $n \times n$ matrix $C = \begin{bmatrix} c_{ij} \end{bmatrix}$ is called the **cofactor matrix** of A.

(Recall that $\tilde{A}_{ij}$ is the matrix obtained from A by deleting the i^{th} row and j^{th} column and is sometimes called the ij^{th} minor matrix of A. $\det \tilde{A}_{ij}$ can be called the ij^{th} **minor** of A or the **minor of the element** a_{ij} of A.)

In this context, a cofactor is sometimes called a signed minor.

<u>Note:</u> c_{ij} is a scalar (real number) but $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix.

We can now restate the definition of determinant in terms of cofactors.

(i) The determinant of a 1 by 1 matrix $\begin{bmatrix} a \end{bmatrix}$ is a.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} a_{1j}\, c_{1j} = a_{11}\, c_{11} + a_{12}\, c_{12} + \cdots + a_{1n}\, c_{1n}$$

where c_{ij} is the cofactor of a_{ij}.

The transpose of the cofactor matrix C of A is called the **classical adjoint** of A and denoted by adj A ;i.e., adj $A = C^T$.

Theorem: If A is any square matrix, then

$$A\,(\,\mathrm{adj}\,A\,) = (\,\det A\,)\,I = (\,\mathrm{adj}\,A\,)\,A\,.$$

In particular, if $\det A \neq 0$, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A$$

where adj $A = C^T$, C being the cofactor matrix of A.

Cramer's Rule If A is an invertible $n \times n$ matrix, the solution to the system

$$A\,X = B$$

of n equations in the variables $x_1,\ x_2,\ \cdots,\ x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A},\quad x_2 = \frac{\det A_2}{\det A},\quad \cdots,\quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing the k^{th} column of A by B.