* SOLUTIONS *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

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Date: February 19, 2011
Time: 9:00 am
Duration: 110 minutes

Provide the following information:

(Print) Surname: SOLUTIONS + Basic Stats

(Print) Given Name(s): ______

Student Number: ______

Signature: ______

Tutorial Number (e.g. TUT0033): ______

Carefully circle the name of your Teaching Assistant:

Jaehyun CHO	Zhou (Joe) LI	Pourya MEMARPANAHI
Srishta CHOPRA	Yik Chau (Kry) LUI	Zheng WANG
Yan (Mary) HE	Yiwen (Louis) LUO	Kai YANG

Read these instructions:

1. This test has 11 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and indicate clearly the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard hand-held calculator (graphing facility is permitted). The following are forbidden: laptop computers, Blackberrys, cell-phones, I-Pods, MP-3 players, extra paper, textbooks, or notes.
5. You are encouraged to write in pen or other ink, not pencil. Tests written in pencil will be denied any regrading privilege.

* * * SOLUTIONS * * *

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6
E	B	D	C	E	B

Do not write anything in the boxes below.

Info.	Part A
2	24

Part B

1	2	3	4	5	6
12	13	10	13	14	12

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2**. Each right answer earns 4 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

1. The minimum value of $Z = 3x - 2y$ subject to $0 \le x \le 2$, $y \ge -2$, and $y \le x + 1$ is

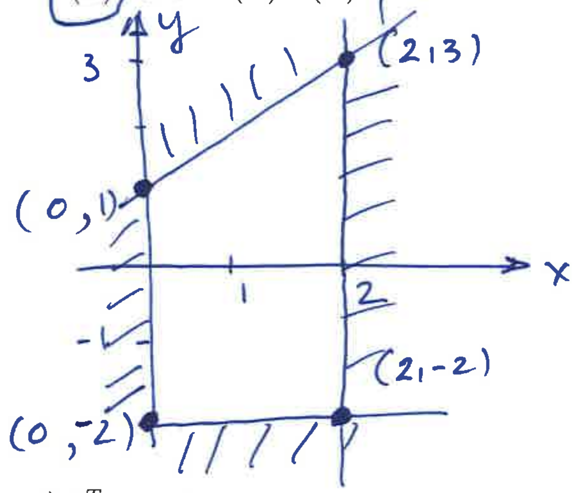
   (A) 4 (B) 2 (C) 0 (D) -1 (E) none of (A) - (D)

$Z(0,1) = -2$ MIN

$Z(2,3) = 0$

$Z(2,-2) = 10$ MAX

$Z(0,-2) = 4$



2. If $A = \begin{bmatrix} 3 & -3 \\ -4 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ then $(A - 2I)B^T$ equals

   (A) $\begin{bmatrix} 11 & 6 \\ -20 & -12 \end{bmatrix}$ (B) $\begin{bmatrix} -11 & -6 \\ 20 & 12 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -2 \\ -4 & -4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ -16 & 12 \end{bmatrix}$

   (E) none of (A) - (D)

$(A - 2I)B^T = \begin{bmatrix} 1 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 20 & 12 \end{bmatrix}$

3. Let $H = [h_{ij}]$ be the 10×10 lower triangular matrix where $h_{ij} = -(i - 4j)^2 - 3$ for all $i \ge j$. The largest element in H is

   (A) -4 (B) -3 (C) 1 (D) 0 (E) none of (A) - (D)

$\because$ H is lower triangular $\Rightarrow h_{ij} = 0$ when $i < j$

For $i \geq j$, $h_{ij} = -(i-4j)^2 - 3 \le -3$

$\therefore$ largest element is 0

4. If $A = \begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix}$ and C is a 2×2 matrix such that $(C^{-1}A)^{-1} = (A^{-1})^2$ then C equals

(A) A (B) $\begin{bmatrix} 5 & -8 \\ -2 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix}$ (D) $-A$ (E) none of (A) - (D)

$$(C^{-1}A)^{-1} = (A^{-1})^2 \Rightarrow A^{-1}C = A^{-1}A^{-1} \Rightarrow C = A^{-1}$$

$$A^{-1} = \begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix}$$

5. Let $Z = ax + by$ where x and y are variables and a and b are constants satisfying $0 < 2a < b$. If (x, y) satisfies the inequalities $x, y \geq 0$ and $2x + 3y \leq 6$ then

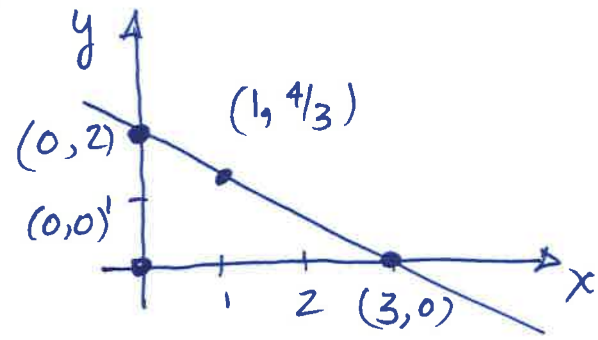
(A) Z has a maximum at $(3, 0)$ or $(0, 2)$

(B) Z has a minimum at $(0, 0)$

(C) Z could have a maximum at the point $(1, 4/3)$

(D) each of (A), (B), and (C) is true

(E) (A) and (B) are true and (C) is false



$Z(3,0) = 3a$ $Z(0,2) = 2b$

$0 < 2a < b \Rightarrow 0 < 3a < \frac{3}{2}b < 2b$

∴ MAX occurs only at (0,2)

∴ (A) ✓

$Z(0,0) = 0$ = MIN ∴ (B) ✓

(C) ✗ because Z is MAXED only at (0,2)

∴ (D) ✗

6. Exactly how many of the following statements are always true?

✗ (i) Every system of $m \geq 2$ linear equations in $n > m$ variables has infinitely many solutions.

✗ (ii) If a square matrix has no zero rows then it is invertible.

✓ (iii) If G is a square matrix of order $n \geq 2$ and $det(G) \neq 0$ then the matrix equation $GX = 0$ has only the trivial solution.

✗ (iv) A non-zero linear objective function defined on a non-empty feasible region has a maximum value, a minimum value, or possiblly both.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

(Be sure you have printed the Multiple Choice answers in the boxes on page 2)

Part B - Full Solution Problem Solving

1. Let $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

(a) Use the method of row reduction to determine whether A is invertible. If it is, find A^{-1}. If it is not, then briefly justify why this is the case. Show all of your work and correct notation for row operations. [10 points]

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$\because$ A reduces to I,

A is invertible.

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Use your answer to (a) to determine whether the matrix equation $AX = B$ has a solution for every 3×1 matrix B. [2 points]

Yes we do have a solution. Let B be a 3×1 matrix

If $AX = B$

then $A^{-1}(AX) = A^{-1}B \Rightarrow IX = A^{-1}B$

$\therefore X = A^{-1}B$ is uniquely determined given B.

2. Find the maximum and minimum values of the function $Z = 6x - 7y$ subject to the constraints: $x + 2y \geq 0$, $x + y \leq 5$, $x - y \leq 3$, $x \geq 0$, and $y \leq 3$. Your solution should clearly show the feasible region, labeled corner points, the optimum values of Z, and all point(s) where the optimum values occur. [13 points]

Calculations for intersections

$y = 3$ in $x + y = 5$
$\Rightarrow x = 2$ $(2, 3)$ ✓

$x + y = 5$
$x - y = 3$ $(4, 1)$ ✓
$2x = 8 \Rightarrow x = 4, y = 1$

$x - y = 3$
$x + 2y = 0$ $(2, -1)$
$3y = -3 \Rightarrow y = -1, x = 2$

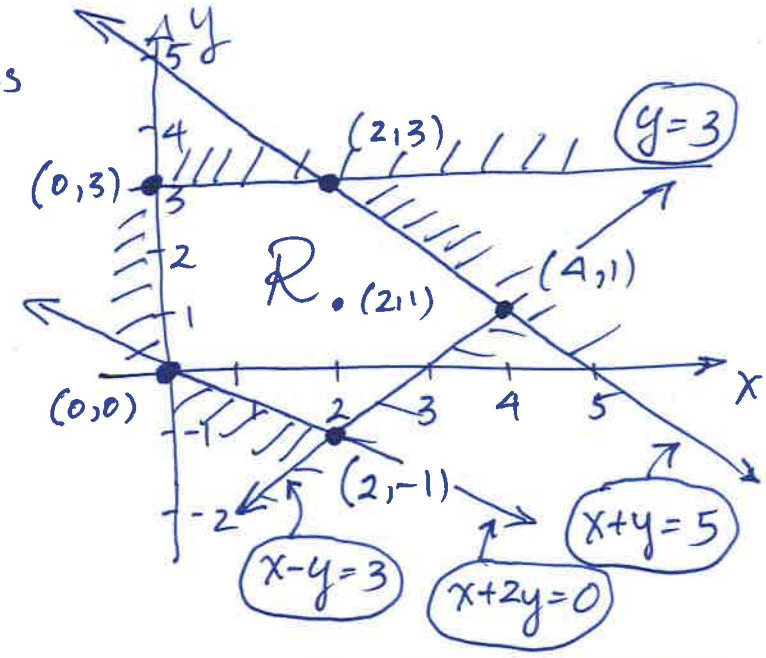
"Test point" $(2, 1)$:

$2 \geq 0$ ✓ $1 \leq 3$ ✓

$2 + 2 \geq 0$ ✓

$2 + 1 \leq 5$ ✓

$2 - 1 \leq 3$ ✓



Feasible region is R.
It is non-shaded above.
It is clearly non-empty.
It is bounded (and lies in the circle of radius 5 centered @ $(0,0)$).

By FTLP, Z is optimized at corner points.

Evaluation @ corner points

$Z(0,0) = 0$
$Z(0,3) = -21$
$Z(2,3) = -9$
$Z(4,1) = 17$
$Z(2,-1) = 19$

By FTLP & evaluation @ left:

MAX value of Z is 19 @ $(2,-1)$

MIN value of Z is -21 @ $(0,3)$

where (x,y) satisfies all constraints.

3. Use matrix reduction to solve the system of linear equations [10 points]

$$\left.\begin{aligned} -x_1 + 2x_2 + x_3 + 0x_4 + x_5 &= 7 \\ 2x_1 - 4x_2 - x_3 + 2x_4 + 0x_5 &= 1 \\ x_1 - 2x_2 + 0x_3 + 2x_4 + 2x_5 &= 5 \end{aligned}\right\} (*)$$

(To obtain full points your answer must show the reduced form of the augmented matrix and all of your work)

Augmented matrix for $(*)$
$$\left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 1 & 7 \\ 2 & -4 & -1 & 2 & 0 & 1 \\ 1 & -2 & 0 & 2 & 2 & 5 \end{array}\right] \sim$$

$-R_1 \to R_1$
$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 0 & -1 & -7 \\ 2 & -4 & -1 & 2 & 0 & 1 \\ 1 & -2 & 0 & 2 & 2 & 5 \end{array}\right] \sim$$

$(R_2 - 2R_1) \to R_2$

$(R_3 - R_1) \to R_3$
$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 0 & -1 & -7 \\ 0 & 0 & 1 & 2 & 2 & 15 \\ 0 & 0 & 1 & 2 & 3 & 12 \end{array}\right] \sim$$

$(R_3 - R_2) \to R_3$
$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 0 & -1 & -7 \\ 0 & 0 & 1 & 2 & 2 & 15 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \sim$$

$(R_1 + R_2) \to R_1$
$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & 1 & 8 \\ 0 & 0 & 1 & 2 & 2 & 15 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \sim$$

$(R_1 - R_3) \to R_1$

$(R_2 - 2R_3) \to R_2$
$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & 0 & 11 \\ 0 & 0 & 1 & 2 & 0 & 21 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

Reduced matrix for $(*)$

<u>Solution:</u>
$x_1 = 2r - 2s + 11$
$x_2 = r$
$x_3 = -2s + 21$
$x_4 = s$
$x_5 = -3$

$r, s \in \mathbb{R}$ are parameters.

4. Suppose you have available up to \$17,000 to invest jointly in two types of shares, A and B. Each A share costs \$6 to purchase and has a predicted revenue upon selling of \$10 per share. Each B share costs \$4 to purchase and has a predicted revenue upon selling of \$7 per share. There are two criteria for buying these shares:

(i) you must buy at least 500 of the B shares and (ii) the number of A shares bought must be at least 75% of the number of B shares bought.

Find the x and y (the number of A and B shares bought in 1,000's, respectively) that you should purchase so as to maximize your predicted profit subject to all of the constraints above.

Remember: Profit = Revenue - Cost.

(To obtain full points your solution should show all details as described in question 2)

[13 points]

$x =$ # of A shares in 1,000's , $x \geq 0$

$y =$ " " B " " " , $y \geq 0$

Total investment amount $\quad 6x + 4y \leq 17$ (1)

(i) $\rightarrow$ $\quad y \geq 1/2$ (2)

(ii) $\rightarrow$ $\quad x \geq \frac{3}{4}y$ (3)

$x, y \geq 0$ (4)

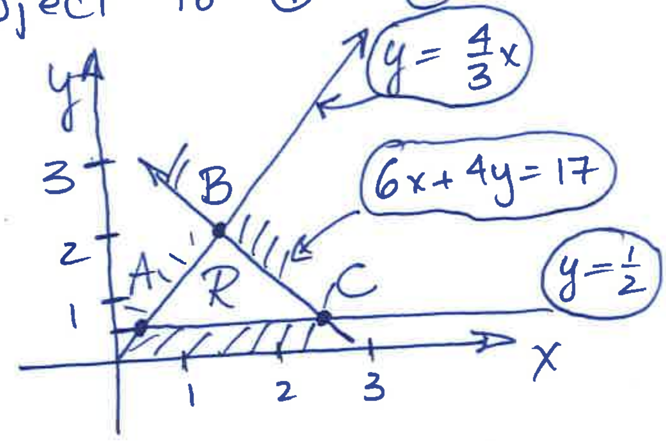
Profit

A: $10 - 6 = 4$ ✓

B: $7 - 4 = 3$ ✓

Profit $\quad P = 4x + 3y$

LP Problem: Maximize P subject to (1) – (4)



Intersection points:

A: $\frac{1}{2} = \frac{4}{3}x \Rightarrow x = \frac{3}{8}$

$A = \left(\frac{3}{8}, \frac{1}{2}\right)$

B: $6x + 4\left(\frac{4}{3}x\right) = 17$

$\frac{34}{3}x = 17 \Rightarrow x = \frac{3}{2}, y = 2$

$B = \left(\frac{3}{2}, 2\right)$

C: $6x + 4\left(\frac{1}{2}\right) = 17$

$x = \frac{5}{2} \quad C = \left(\frac{5}{2}, \frac{1}{2}\right)$

R is non-shaded, bounded, non-empty feasible region.

By FTLP, P is MAXED @ corner point

$P(A) = 4\left(\frac{3}{8}\right) + 3\left(\frac{1}{2}\right) = 3$

$P(B) = 4\left(\frac{3}{2}\right) + 3(2) = 12$

$P(C) = 4\left(\frac{5}{2}\right) + 3\left(\frac{1}{2}\right) = 11\frac{1}{2}$

ANSWER: MAX PROFIT of \$12,000 when $x = 1.5$, $y = 2$

5. A company has two small manufacturing plants, P_1 and P_2. Each plant has the same four utilities: u_1 = electricity, u_2 = water, u_3 = natural gas, u_4 = internet.

Let $C = [c_{i,j}] = \begin{bmatrix} 26 & 3 & 20 & 1.2 \\ 17 & 2 & 18.6 & 3.1 \end{bmatrix}$

where $c_{i,j}$ = the monthly cost in \$100's for utility u_j for plant P_i. For example, plant 1 has a water cost of \$300 a month and plant 2 has an internet cost of \$310 a month.

(a) State the matrices F, A, and B such that

(i) the entries in the product CF give each plant's total monthly utility cost.

(ii) the entries in the product CA^T give, for each plant, the average of the four monthly utility costs.

(iii) the entries in the product BC give, for each utility, the average of the monthly costs over the two plants. [6 points]

$F = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(b) What information does the matrix product $2BCF$ give? [2 points]

$2BCF = 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ = Grand total of all utility costs for both plants.

(c) Suppose we have the following new utility cost information. For both plants: the monthly electricity cost increases by 3%, the monthly natural gas cost decreases by 2%, and the monthly internet cost increases by \$15. There is no change in the monthly water cost for either plant. State the matrix K such that the entries in the sum $K + C$ reflect the new monthly utility costs. [6 points]

3% of 26 = .78 2% of 20 = .4

3% of 17 = .51 2% of 18.6 = .372

\$15 → .15 (in \$100).

$\therefore K = \begin{bmatrix} .78 & 0 & -.4 & .15 \\ .51 & 0 & -.372 & .15 \end{bmatrix}$

6. (a) Let $E = \begin{bmatrix} 1 & 2 & 0 \\ t & t & -1 \\ 2 & 8 & t \end{bmatrix}$ where $t \in \mathbf{R}$. Find all values of t for which E is invertible.

[6 points]

E is invertible iff $\det(E) \neq 0$

$$\det(E) = 1\det\begin{bmatrix} t & -1 \\ 8 & t \end{bmatrix} - 2\det\begin{bmatrix} t & -1 \\ 2 & t \end{bmatrix}$$

$$= t^2 + 8 - 2(t^2+2)$$

$$= -t^2 + 4$$

$\det(E) = 0 = -t^2+4 \Longleftrightarrow t = \pm 2$

$\therefore$ E is invertible when $t \neq \pm 2$, $t \in \mathbb{R}$

(b) Let $n \geq 3$ and let J be the $n \times n$ matrix all of whose entries are 1.

Prove that $(I - J)^{-1} = (I - \frac{1}{n-1} J)$

[6 points]

$J = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ $(I - \frac{1}{n-1} J)$ is clearly an $n \times n$ matrix

$J^2 = nJ$

$$(I-J)(I - \frac{1}{n-1}J)$$

$$= I - \frac{1}{n-1}J - J + \frac{1}{n-1}J^2$$

$$= I - \frac{1}{n-1}J - J + \frac{n}{n-1}J$$

$$= I + \left(\frac{-1 - n + 1 + n}{n-1}\right)J = I + \frac{0}{n-1}J = I$$

That is enough to show $(I-J)^{-1} = (I - \frac{1}{n-1}J)$

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Basic Test Stats

Average ≈ 64%

Pass % ≈ 82.8% (i.e. 82.8% of all A33 students passed... very well done)

Number who wrote test = 478

100	0
90's	2.1%
80's	11.5%
70's	22.4%
60's	26.0%
50's	21.0%
40's	13.2%
30's	2.7%
20's	1.3%
10's	0
1's	0

(% are rounded, so may not add to exactly 100%)