* SOLUTIONS *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiners: R. Buchweitz
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Date: March 2, 2015
Time: 5:00 pm
Duration: 110 minutes

Provide the following information:

Surname (PRINT BIG): SOLUTIONS

Given Name(s) (PRINT BIG): ______________________

Student Number: ______________________

Signature: ______________________

Circle the name of your Teaching Assistant and Tutorial Number:

Danny CAO 25

Michelle (Xiaopeng) CUI 19 24

Rui GAO 2 26

Terry (Yaodong) GAO 15

Sherveen GHOFRANI 5

Namhee KANG 7 20

Aaron (Xin) SITU 4 6

Huiyi WANG 12 23

Binya XU 13 16

Elaine (Mengnan) ZHU 1 8

Instructions:

1. This test has 11 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and indicate clearly the location of your continuing work.
3. The following are forbidden at your workspace during any part of the test: calculators, smart phones, tablet devices, any kind of electronic transmission or receiving device, electronic dictionaries, extra paper, textbooks, notes, opaque (non-see through) pen/pencil cases, or food. You may have one drink, but it cannot be in a paper cup or box.
4. You are encouraged to write your test in pen or other ink, not pencil. If any portion of your test is written in pencil, you will be denied any regrading privilege.

* SOLUTIONS *

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
D	A	B	C	C	A	A

Do not write anything in the boxes below.

Info	**Part A**
3	21

Part B

1	2	3	4	5	6
14	14	14	10	11	13

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of Page 2.** Each right answer earns 3 points and no answer or wrong answers earn 0 points. Justification is not required.

1. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then $BA - 3A^T$ equals

(A) $\begin{bmatrix} -2 & 12 \\ -5 & 8 \end{bmatrix}$ (B) $\begin{bmatrix} -2 & -20 \\ 0 & 8 \end{bmatrix}$ (C) $\begin{bmatrix} -2 & 20 \\ 5 & 8 \end{bmatrix}$ (D) $\begin{bmatrix} -2 & 20 \\ -5 & 8 \end{bmatrix}$

(E) none of (A) - (D)

$$BA - 3A^T = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 17 \\ -2 & 29 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 3 & 21 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 20 \\ -5 & 8 \end{bmatrix}$$

2. If A and B are as in Question 1 then the value of $det\left(2AB - B\right)$ is

(A) 430 (B) 33 (C) 350 (D) 290 (E) none of (A) - (D)

$$det(2AB - B)$$
$$= det(2AB - IB)$$
$$= det((2A - I)B)$$
$$= det(2A - I) \cdot det(B)$$
$$= det\left(\begin{pmatrix} 4 & 2 \\ -2 & 14 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot (12 - 2)$$
$$= det\begin{pmatrix} 3 & 2 \\ -2 & 13 \end{pmatrix} \cdot (10)$$
$$= (39 + 4) \cdot (10) = 430$$

3. Let $k < 0$ be a constant. The minimum value of $Z = kx - y$ subject to the inequalities $0 \leq x \leq 2$ and $-1 \leq y \leq x + 2$ is

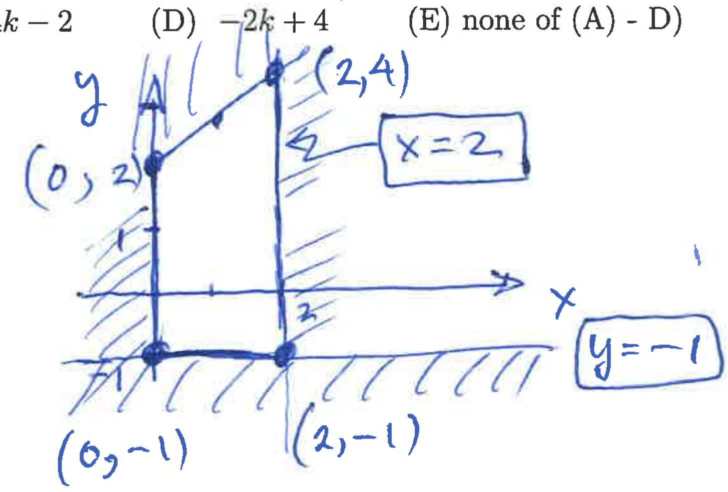
(A) $-2k - 4$ (B) $2k - 4$ (C) $4k - 2$ (D) $-2k + 4$ (E) none of (A) - D)

$Z(0,-1) = 1$

$Z(0,2) = -2$

$Z(2,4) = 2k - 4$

$Z(2,-1) = 2k + 1$



$\therefore k < 0 \Rightarrow 2k - 4$ is the minimum value

4. If G and H are $n \times n$ matrices ($n \geq 3$) such that $det(G) = 6$ and $det(H) = 3$ then the value of $det\left(2G^{-1}H^2\right)$ is

(A) n (B) $3(2^n)$ (C) $3(2^{n-1})$ (D) $3n$ (E) none of (A) - (D)

$$det(2G^{-1}H^2) = 2^n det(G^{-1})(det(H))^2$$
$$= 2^n (det(G))^{-1}(3)^2 = 2^n\left(\frac{1}{6}\right)9 = \frac{3}{2}\cdot 2^n = 3(2^{n-1})$$

5. Let n be an integer, $n \geq 2$. Exactly how many of the following statements are equivalent to the statement: "The $n \times n$ matrix M is invertible."

- $det(M) \neq 0$. ✓
- $MX = 0$ has the trivial solution. ✗ ("only" the trivial sol$^{\underline{n}}$) ✓
- M^k is invertible for some integer $k \geq 2$. ✓
- $MC = CM$ for some $n \times n$ invertible matrix C. ✗ ($MC = CM = I_n$)

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

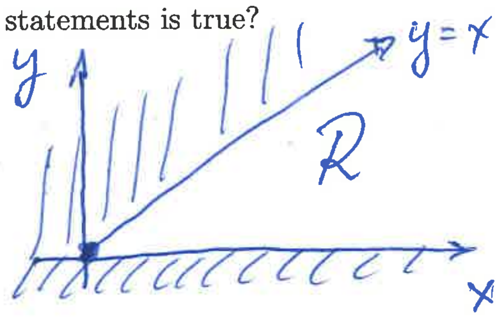
6. Let $\mathcal{R}$ represent the region consisting of all points (x, y) such that $y \geq 0$ and $y \leq x$. Let $Z = -3x + 3y$ where $(x, y) \in \mathcal{R}$. Which one of the following statements is true?

(A) Z has a maximum on $\mathcal{R}$ but no minimum.

(B) Z has a minimum on $\mathcal{R}$ but no maximum.

(C) Z has both a maximum and a minimum on $\mathcal{R}$.

(D) Z has neither a maximum nor a minimum on $\mathcal{R}$.



No minimum: $Z(x,0) = -3x \to -\infty$ as $x \to \infty$

Maximum: Level curve is $-3x + 3y = k$ for $(x,y) \in R$
$\Rightarrow y = x - k/3$ ... slope $= 1$, y-int $= -\frac{k}{3} \leq 0$
Largest k value is $k = 0$.

7. Let $P = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & -5 \\ 3 & \alpha^2 \end{pmatrix}$ where α is a real constant. For which value(s) of α (if any) do P and Q commute?

(A) $\pm\sqrt{5}$ (B) 1 (C) $\pm\sqrt{3}$ (D) ± 5

(E) numbers that are not in (A) - (D) (F) no values of α

$$PQ = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ 3 & \alpha^2 \end{pmatrix} = \begin{pmatrix} 23 & -10 + 5\alpha^2 \\ -9 & 15 + \alpha^2 \end{pmatrix}$$

$$QP = \begin{pmatrix} 4 & -5 \\ 3 & \alpha^2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 23 & 15 \\ 6 - 3\alpha^2 & 15 + \alpha^2 \end{pmatrix}$$

We need $-10 + 5\alpha^2 = 15 \Rightarrow -9 = 6 - 3\alpha^2$

(Be sure you have printed the Multiple Choice answers in the boxes on Page 2)

$\therefore 5\alpha^2 = 25$ $3\alpha^2 = 15$

$\alpha^2 = 5$ $\alpha^2 = 5$

$\therefore \alpha = \pm\sqrt{5}$

Part B - Full Solution Problem Solving Put your solutions and rough work in the answer spaces provided. Full-marks are awarded for solutions that are correct, complete, and show a sufficient amount of relevant concepts from MATA33.

1. In all of this question, let $A = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{pmatrix}$.

   (a) Find the inverse of A and check your answer by multiplication. [10 points]

$$A^{-1} = \begin{pmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \\ 4 & 5 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 1 & -4 & 4 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -12 & -3 & 4 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 11 & 3 & -4 \\ 0 & 1 & 0 & -12 & -3 & 4 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right)$$

$$\sim \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{array}}_{A^{-1}}\right)$$

   (b) Express the matrix $B = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ as a sum of scalar multiples of the columns of A. [4 points]

Consider $AX = B \Rightarrow X = A^{-1}B$

$$X = \begin{pmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 15 + 8 + 25 \\ -12 - 6 + 8 \\ -4 - 2 - 5 \end{pmatrix}$$

$$= \begin{pmatrix} 48 \\ -38 \\ -11 \end{pmatrix}$$

$$\therefore B = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = (48)\begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + (-38)\begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} + (-11)\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

2. For each of the following systems of linear equations, use the method of reduction to find the solution(s) or determine that the system is inconsistent. If the system is consistent, be sure to display the reduced form of its augmented matrix.

(a)

$$\begin{aligned} 3x + 2y + 6z &= 7 \\ x - y + z &= 1 \\ x + 4y + 4z &= 0 \end{aligned}$$

[6 points]

$$\left(\begin{array}{ccc|c} 3 & 2 & 6 & 7 \\ 1 & -1 & 1 & 1 \\ 1 & 4 & 4 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 3 & 2 & 6 & 7 \\ 1 & 4 & 4 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 5 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

Row 3 shows system is INCONSISTENT!

(b)

$$\begin{aligned} x_1 + 2x_2 - 5x_3 + 6x_4 + 14x_5 &= 15 \\ 2x_1 + 4x_2 - 9x_3 + 12x_4 + 28x_5 &= 41 \\ -x_1 - 2x_2 + 5x_3 - 5x_4 - 12x_5 &= -15 \end{aligned}$$

[8 points]

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 15 \\ 2 & 4 & -9 & 12 & 28 & 41 \\ -1 & -2 & 5 & -5 & -12 & -15 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 15 \\ 0 & 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 6 & 14 & 15 \\ 0 & 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\sim \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 70 \\ 0 & 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

This matrix is reduced.

Basic: x_1, x_3, x_4

Free: x_2, x_5

Solution:

$x_1 = -2r - 2s + 70$

$x_2 = r$

$x_3 = 11$

$x_4 = -2s$

$x_5 = s$

$r, s \in \mathbb{R}$ are parameters

3. Find the maximum and minimum values (and all points(s) where they occur) of the objective function $Z = -3x + 6y$ for the feasible region $\mathcal{R}$ that is given by the five constraints:

$$x - 4y + 15 \geq 0, \qquad x \leq 5, \qquad x - 2y \leq 5, \qquad x + 3y \geq 0, \qquad y \leq 4x$$

(To earn full points, your solution must include a neat, labeled diagram of the feasible region $\mathcal{R}$, and all calculations/justifications.)

[14 points]

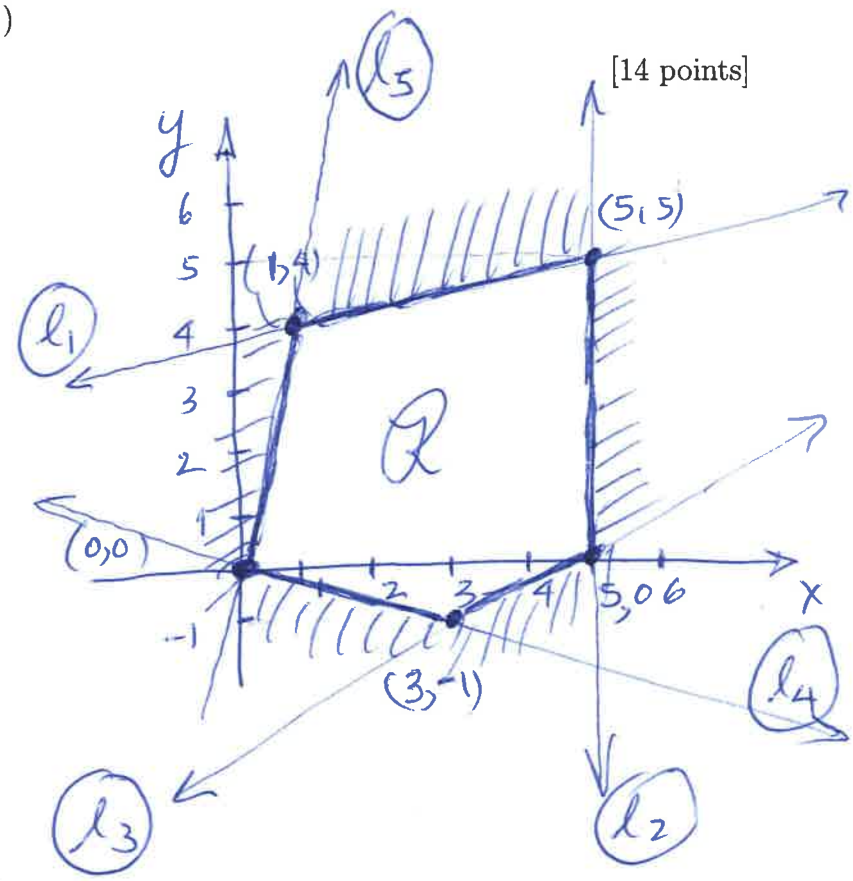
l_1 $x - 4y + 15 = 0$
Points: $(1,1)$, $(5,5)$

l_2 $x = 5$
Points: $(5,5)$, $(5,0)$

l_3 $x - 2y = 5$
Points: $(3,-1)$, $(5,0)$

l_4 $x + 3y = 0$
Points: $(0,0)$, $(3,-1)$

l_5 $y = 4x$
Points: $(0,0)$, $(1,4)$



$\mathcal{R}$ = feasible region
$= \{(x,y) \in \mathbb{R}^2 \mid$ all inequalities are satisfied $\}$

$\mathcal{R}$ is non-empty, bounded, and standard (i.e. all corners & line segments $\in \mathcal{R}$)

By FTLP, Z is optimized at corners of $\mathcal{R}$.

Corners	Evaluation
$(0,0)$	$Z(0,0) = 0$
$(1,4)$	$Z(1,4) = -3+24 = 21$
$(5,5)$	$Z(5,5) = -15+30 = 15$
$(5,0)$	$Z(5,0) = -15$
$(3,-1)$	$Z(3,-1) = -9-6 = -15$

MAX $= 21$ @ $(1,4)$
MIN $= -15$ @ all points on segment joining $(5,0)$ to $(3,-1)$

4. The parts of this question are independent of each other.

(a) Let n be an integer, $n \geq 4$. If S is an $n \times n$ symmetric matrix, what is the maximum number (in terms of n) of distinct entries that S can have? [5 points]

Answer is $\frac{n(n+1)}{2} = 1+2+3+\cdots+n$.

Here's why: $\because$ S is symmetric we have $S^T = S$ $\therefore$ $s_{ij} = s_{ji}$ whenever $i \neq j$.

$\therefore$ the only possible distinct entries are s_{ij} where $i \leq j$ (upper Δ ar part of S).

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2n} \\ \vdots & \vdots & & & \\ s_{n1} & s_{n2} & \cdots & & s_{nn} \end{pmatrix}$$

Now count the possibilities to get: $1+2+3+\cdots+n = \underbrace{\frac{n(n+1)}{2}}$ (MATH32 formula)

(b) Show that the matrix $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ is never invertible no matter what the values of the constants a, b and c are. [5 points]

Solution #1 $\det\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = (-a)\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix}$

$= (-a)(bc) + b(ac) = 0$

$\therefore$ matrix can never be invertible.

Solution #2: Let A = matrix above.

Note that A is skew symmetric ... $A^T = -A$

$\therefore \det(A) = \det(A^T) = \det(-A) = -\det(A)$

$\Rightarrow 2\det(A) = 0$ so $\det(A) = 0$

$\therefore$ A can never be invertible.

5. The following is information about an outdoor adventure business in Northern Ontario. Suppose you rent "boats" (i.e. canoes and kayaks) to people to travel 10km down a scenic river. You determine that $45,000 is available to you to buy new boats. A kayak costs $600 to buy and a canoe costs $750 to buy. Your storage facility up north can hold up to 65 boats. Each canoe bought will rent for $25 per day and each kayak bought will rent for $30 a day. How many canoes and kayaks should you buy to earn the most revenue per day? What is the maximum revenue? Formulate this problem as a Linear Programming Problem and solve it. Produce an accurate, labeled feasible region and show all appropriate details.

[11 points]

Let x represent the number of canoes and y represent the number of kayaks.

We have constraints (i.e. inequalities)

(1) $x + y \le 65$ (storage)

(2) $750x + 600y \le 45{,}000$ (funds available)

(3) $x, y \geq 0$ (non-negativity)

Revenue f^n (i.e. objective f^n) is

$Z = 25x + 30y$

Linear Programming Problem: maximize Z subject to the inequalities above

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Re-write (2) as $5x + 4y \le 300$

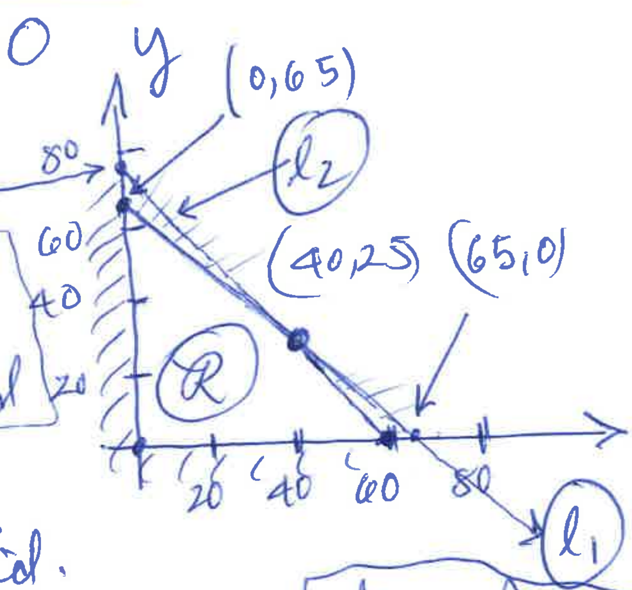
l_1: $x + y = 65$

l_2: $5x + 40y = 300$

$l_1 \cap l_2$: $x = 40,\ y = 25$

R bounded, non-empty, standard

(FTLP is valid.



R has corners | Eval

(0,0) $Z(0,0) = 0$

(0,65) $Z(0,65) = 1950$

(40,25) $Z(40,25) = 1000 + 750 = 1750$

(60,0) $Z(60,0) = 1500$

Max Rev $= 1950$

$x = 0$ (canoes)

$y = 65$ (kayaks)

6. The parts of this question are independent of each other.

(a) Find all real values of x such that $\underbrace{det\begin{pmatrix} x & -1 \\ 3 & 1-x \end{pmatrix}}_{f(x)} = \underbrace{det\begin{pmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{pmatrix}}_{g(x)}$. [8 points]

$f(x) = x(1-x)+3 = -x^2+x+3$

$g(x) = \det\begin{pmatrix} x & -6 \\ 3 & x-5 \end{pmatrix} - 3\det\begin{pmatrix} 2 & x \\ 1 & 3 \end{pmatrix}$

$= x(x-5)+18-3(6-x)$

$= x^2-5x+18-18+3x$

$= x^2-2x$

Equate, solve : $f(x) = g(x)$

$-x^2+x+3 = x^2-2x$

$-2x^2+3x+3=0 \rightarrow x = \frac{-3\pm\sqrt{9-4(-2)(3)}}{-4}$

$= \frac{-3\pm\sqrt{9+24}}{-4}$

$= \frac{-3\pm\sqrt{33}}{-4} = \frac{3\mp\sqrt{33}}{4}$

(b) Assume A and B are $n\times n$ matrices ($n\geq 2$) such that A and $A+B$ are invertible. Prove that the matrix $I+BA^{-1}$ is invertible. [5 points]

Proof: A is invertible $\Rightarrow A^{-1}$ exists and is invertible $((A^{-1})^{-1}=A)$

Multiply : $(A+B)A^{-1} = AA^{-1}+BA^{-1}$

$= I + BA^{-1}$

$\therefore$ $I+BA^{-1}$ = product of the invertible matrices $A+B$ and A^{-1}

$\therefore$ $I+BA^{-1}$ is invertible because products of invertible $n\times n$ matrices are invertible.