*** SOLUTIONS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: June 14, 2014
Time: 5:00 pm
Duration: 120 minutes

(Print CAPITALS) LAST NAME: ______________________

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: SOLUTIONS & BASIC STATS

Circle the name of your Teaching Assistant and Tutorial Number:

Pourya MEMARPANAHI 3 6

1 2 Zhaoyun (Helen) WANG

Allan MENEZES 4

Read these instructions:

1. This test has 11 pages. It is your responsibility to check at the beginning of the test that all of these 11 pages are included.
2. If you need extra answer space for any question, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. The following are forbidden at your workspace: calculators, any other kind of electronic aid or device (e.g. cell/smart phones, i-pads, i-phones, etc.), scrap paper, food, textbooks, bags, pencil/pen carrying cases, drinks in a paper cup or box or similar container that has a removable label.
4. Cell/smart/i-phones must be turned off and left at the front of the test room.
5. You are encouraged to write in pen or other ink, not pencil. If any part of your test is written in pencil, then you will be denied any re-grading opportunity.

— SOLUTIONS —

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
D	D	B	E	E	D	B

Do not write anything in the boxes below.

2 points ⟺
all information
requested is
on the cover page 1.

Info.	Part A
2	21

Part B

1	2	3	4	5	6
16	23	9	8	16	5

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 3 points and no answer/wrong answer/ambiguous answers earn 0 points. A small workspace is provided for your calculations and rough work.

1. If $A = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ and $B = [B_{ij}] = AC^{-1} + C$ then $B_{12} - B_{21}$ is

   (A) -23 (B) -3 (C) 33 (D) 31 (E) 3 (F) none of (A) - (E)

$$C^{-1} = \frac{1}{5-4}\begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix}$$

$$AC^{-1} + C = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 3 \\ -31 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9 & 4 \\ -27 & 12 \end{bmatrix} = B$$

$$B_{12} - B_{21} = 4 - (-27) = 31$$

2. If A and C are as in Question 1 above, then the value of $det(5AC)$ is

   (A) 115 (B) -35 (C) -175 (D) 575 (E) none of (A) - (D)

$$\det(5AC) = 5^2 \det(A)\det(C)$$
$$= 25(8+15)(5-4)$$
$$= 25(23)$$
$$= 575$$

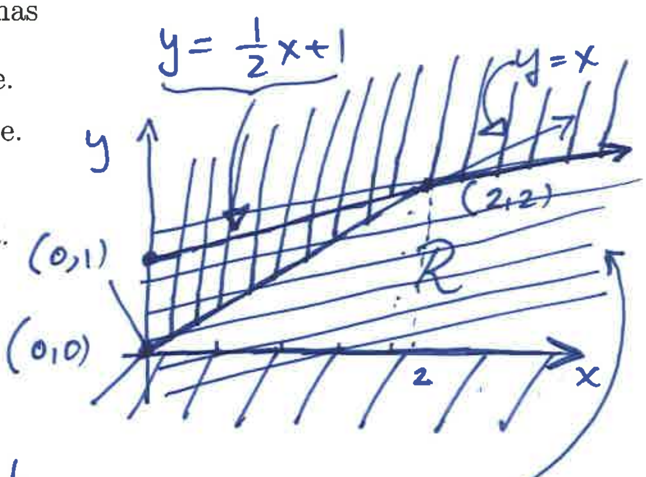
3. For the feasible region $\mathcal{R}$ determined by the constraints $0 \le y \le x$ and $2y \le x + 2$ we may conclude that the objective function $Z = 3x - 4y$ has

   ✗ (A) A maximum value on $\mathcal{R}$, but no minimum value.

   ✓ (B) A minimum value on $\mathcal{R}$, but no maximum value.

   ✗ (C) Both a maximum and a minimum value on $\mathcal{R}$.

   ✗ (D) Neither a maximum nor a minimum value on $\mathcal{R}$.



Level curves for varying k-values.

Z has <u>no maximum</u>:

$Z(x,0) = 3x \longrightarrow \infty$ on x-axis

$\therefore$ (A) ✗ and (C) ✗

Level curve is $3x - 4y = k$

$4y = 3x - k$

$\therefore y = \frac{3}{4}x - \frac{k}{4}$

"The highest" level curve touching $\mathcal{R}$ does so @ $(2,2)$ and there $k = -2$ $\therefore$ Z has a minimum

4. Let $U = [u_{ij}]$ be the 10×10 upper triangular matrix such that $u_{ij} = i^2 - j^2 + 2$. The largest element in U is

(A) 0 (B) 1 (C) 19 (D) 101 (E) 2 (F) 102 (G) none of (A) - (F)

$\because$ U is upper $\Delta^{ar} \Rightarrow u_{ij} = 0$ when $i > j$

When $i < j$ we have $u_{ij} = i^2 - j^2 + 2 < 2$

But when $i = j$, $u_{ij} = i^2 - j^2 + 2 = 2$ $\therefore$ largest $= 2$

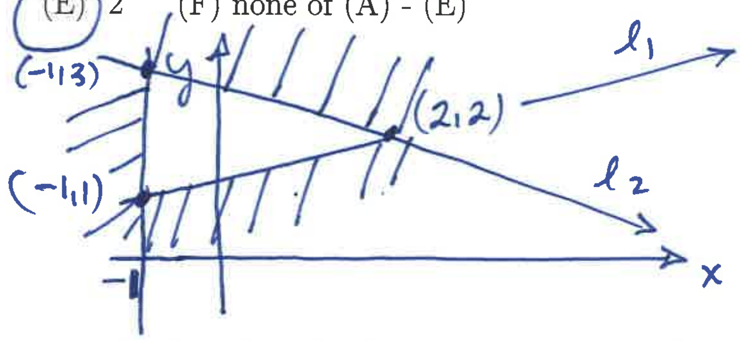
5. The maximum value of the objective function $Z = 2x - y$ satisfying $\underbrace{\frac{1}{3}x + \frac{4}{3}}_{\ell_1} \leq y \leq \underbrace{-\frac{1}{3}x + \frac{8}{3}}_{\ell_2}$ and $x \geq -1$ is

(A) -6 (B) -2 (C) -3 (D) 3 (E) 2 (F) none of (A) - (E)

$Z(-1,3) = -2 - 3 = -5$

$Z(-1,1) = -2 - 1 = -3$

$Z(2,2) = 4 - 2 = 2 = \text{MAX}$



6. Assume S is a 2×2 invertible symmetric matrix. Which of the following are also symmetric?

(i) $-S$ (ii) S^{-1} (iii) SR where R is a 2×2 invertible symmetric matrix

(A) (i) only (B) (ii) only (C) (iii) only (D) (i) and (ii)

(E) (i) and (iii) (F) (ii) and (iii) (G) all of (i), (ii), and (iii)

$(-S)^T = -(S^T) = -S$ $\therefore$ (i) ✓

$(S^{-1})^T = (S^T)^{-1} = S^{-1}$ $\therefore$ (ii) ✓

$(SR)^T = R^T S^T = RS$ but SR need not $= RS$

$\therefore$ (iii) ✗

7. Exactly how many of the following statements are always true?

✓ • If A is a square matrix of order $n \geq 2$ and the homogeneous system $AX = 0$ has only the trivial solution, then A must be invertible.

✗ • A non-zero linear objective function defined on a non-empty, standard feasible region has a maximum value, or a minimum value, or perhaps both. Bounded?

✓ • If A and B are 2×2 equivalent matrices and the reduced form of A is I, then B is invertible.

✗ • If a non-zero linear objective function Z has the same value at two different corner points P and Q of non-empty, bounded, standard feasible region, then Z is optimized at all points on the line segment joining P and Q. Is Z optimal @ P & Q?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 0

*** DID YOU PUT THE ANSWERS IN THE BOXES ON PAGE 2? ***

Part B - Full Solution Problem Solving Full points will be awarded for your solutions if and only if they are correct, complete, and show sufficient relevant concepts form MATA33.

1. In all of this question let $A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

(a) Find A^{-1} by the method of row-reduction. [11 points]

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & 5 & 0 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & -5 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 & 13 & -3 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 6 & -14 & 3 \\ 0 & 1 & 0 & -5 & 13 & -3 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 16 & -40 & 9 \\ 0 & 1 & 0 & -5 & 13 & -3 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{pmatrix}$$

(It's a good idea to check this by multiplication)

(b) Use your answer in (a) to solve the equation $AX = B$ where $B = \begin{pmatrix} 2 \\ -3 \\ -17 \end{pmatrix}$ [5 points]

For $AX = B$,

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$\therefore X = A^{-1}B = \begin{pmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -17 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

(again, a good idea to check by multiplication)

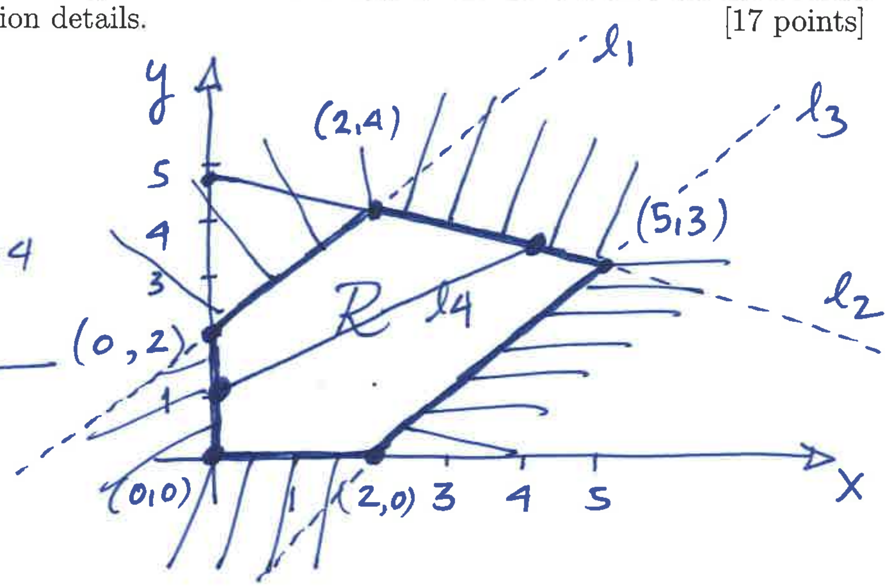
2. (a) Find the maximum and minimum values of the objective function $Z = 5x + 6y$ subject to the five constraints:

$$x \geq 0 \qquad y \geq 0 \qquad \underbrace{-x + y \leq 2}_{l_1} \qquad \underbrace{x + 3y \leq 14}_{l_2} \qquad \underbrace{x - y \leq 2}_{l_3}$$

Your answer should include a neat, labeled feasible region, the location of all optimizing points, and sufficient calculation details. [17 points]

$l_1 \cap l_2$: $-x+y=2$ l_1
$x+3y=14$ l_2

Add: $4y=16 \therefore y=4$
$\therefore x=2$ $(2,4)$



$l_3 \cap l_2$: $x+3y=14$
$x-y=2$

Sub: $4y=12 \therefore y=3$
$\therefore x=5$ $(5,3)$

R is $\neq \phi$, bounded, standard
$\therefore$ by FTLP Z is optimized at corner points.

Corner points of R:
$(0,0)$, $(0,2)$, $(2,4)$, $(5,3)$, and $(2,0)$

$Z(0,0)=0$
$Z(0,2)=12$
$Z(2,4)=10+24=34$
$Z(5,3)=25+18=43$
$Z(2,0)=10$

$\therefore$ Max value of Z is 43 @ $(5,3)$
Min value of Z is 0 @ $(0,0)$

(b) Find the maximum and minimum value of the objective function in part (a) where (x, y) satisfies all constraints in part (a) plus the extra constraint $\underbrace{-7x + 12y = 12}_{l_4}$. [6 points]

When $x=0$ in l_4, $y=1$ $(0,1)$

l_4 re-arranged is $y=\frac{7}{12}x+1$
$\therefore l_4$ intersects l_2

$l_4 \cap l_2$: $-7x+12y=12$
$x+3y=14$
$-11x=-44$
$\therefore x=4$
$y=\frac{10}{3}$
Point is $(4,\frac{10}{3})$

$l_4 \cap R$ is the segment with end points $(0,1)$ and $(4,\frac{10}{3})$
By FTLP we evaluate:
$Z(0,1)=6$
$Z(4,\frac{10}{3})=20+20=40$
$\therefore$ Max $=40$ @ $(4,\frac{10}{3})$
Min $=6$ @ $(0,1)$

3. The parts of this question are independent of each other.

(a) Find all possible diagonal matrices E such that $E^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. [3 points]

There are four matrices E:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Find all real numbers t such that the matrix $A - tI$ is <u>not invertible</u>. [6 points]

Consider: $\det(A - tI)$

$$= \det\left(\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1-t & 3 \\ 4 & 2-t \end{pmatrix}$$

$$= (1-t)(2-t) - 12$$

$$= 2 - 3t + t^2 - 12$$

$$= t^2 - 3t - 10 = (t-5)(t+2)$$

$A - tI$ is <u>not</u> invertible $\Leftrightarrow \det(A - tI) = 0$

$\Leftrightarrow t = 5$ or $t = -2$

4. A work shop makes and sells two kinds of decorative iron sculptures: X and Y. Each one of X takes 3 hours to make and 1/2 hour to paint. Each one of Y takes 2 hours to make and 1 hour to paint. There are 240 worker-hours per day available for assembly (i.e. for making X, or Y, or both) and 80 worker-hours per day for painting (i.e. for painting X, Y, or both). The revenue for selling each X is \$50 and for each Y is \$40. The object is to determine the number of X and Y made and sold each day that maximizes the total daily revenue.

Set up the complete linear programming problem that describes the situation above.

You only need to set up the complete linear programming problem. Do not produce the feasible region or solve the problem !

[8 points]

Rough work: x = number of X made & sold / day
y = " " Y " " " / "

Revenue (total) = "X-revenue" + "Y-revenue"
$= 50x + 40y$

Need $x, y \geq 0$

Table:

	X	Y	available
assemble	3	2	240
paint	1/2	1	80

LPP is:

Maximize $Z = 50x + 40y$

subject to

$$3x + 2y \leq 240$$
$$\tfrac{1}{2}x + 1y \leq 80$$
$$x, y \geq 0$$

x, y are as above.

5. (a) Use the method of reduction to solve the homogeneous system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 &= 0 \\ -x_1 + 2x_2 - 4x_3 + x_4 &= 0 \\ 3x_1 \qquad\qquad\quad - 3x_4 &= 0 \end{aligned}$$

Show all work and display the reduced form of the coefficient matrix. [10 points]

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 3 & -6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 3 & -6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The final matrix shows that x_1 and x_2 are basic and x_3 and x_4 are free.

$\therefore$ solution to $AX = O$ is

$x_1 = t$

$x_2 = 2s$

$x_3 = s$

$x_4 = t$ $\qquad$ $t, s \in \mathbb{R}$ are parameters.

(b) Let A be the coefficient matrix in (a). Find $det(A)$. Your answer in part (a) should be helpful.

[3 points]

$\because AX = O$ has infinitely many solutions, the coefficient matrix A is not invertible.

$\therefore \det(A) = 0$.

(Question 5 continued).

(c) Assume that $Y = [6\ \ 2\ \ 0\ \ 3]^T$ is a solution to the matrix equation $AX = B$ where $B = [1\ \ 8\ \ 1\ \ 9]^T$. Use this fact and your answer in (a) to find all solutions to $AX = B$.

[3 points]

Suppose Z is an arbitrary solution to $AX=B$.

(i.e. $AZ=B$.) $\because$ $AY=B$, we have

$AZ=AY$ so $A(Z-Y)=0$. $\therefore$ $Z-Y$ is

an arbitrary solution to the homogeneous system

in part(a). $\therefore$ $Z-Y=\begin{pmatrix} t \\ 2s \\ s \\ t \end{pmatrix}$

$\therefore$ $Z= Y+\begin{pmatrix} t \\ 2s \\ s \\ t \end{pmatrix}$, $s,t \in \mathbb{R}$, gives all solutions to $AX=B$.

6. Let A be an $n \times n$ matrix, $n \geq 3$, such that the matrix equation $AX = B$ has a solution for each $n \times 1$ matrix B. Show that A is invertible. (Hint: Show there is a matrix C such that $AC = I$).

[5 points]

For $k=1,2,3,\ldots,n$, let $E_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$ place

$\therefore$ E_k is the $n\times 1$ column matrix such that

$$(E_k)_{i1} = \begin{cases} 0 & i \neq k \\ 1 & i = k. \end{cases}$$

Let C_k be a solution to $AX=E_k$, $k=1,2,\ldots,n$

$\therefore$ C_k is an $n\times 1$ column matrix and $AC_k=E_k$.

Let C be the $n\times n$ matrix whose k^{th} column is C_k $\therefore$ $C=\begin{bmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_n \\ | & | & & | \end{bmatrix}$ ($n\times n$ matrix)

$$AC = \underbrace{\begin{bmatrix} | & | & & | \\ AC_1 & AC_2 & \cdots & AC_n \\ | & | & & | \end{bmatrix}}_{n\times n} = \begin{bmatrix} | & | & & | \\ E_1 & E_2 & \cdots & E_n \\ | & | & & | \end{bmatrix} = I$$

This shows A is invertible.

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Basic Statistics*

$N = 115$ = # of students who wrote test

$\bar{x} \approx 63.\%$ = test average

- About 86.1% of 115 students got a score of $\geqslant 50\%$
- About 56.5% of 115 students got a score of $\geqslant 60\%$
- About 22.6% of 115 students got a score of $\geqslant 70\%$

	~ % of 115 students
90's	0
80's	7.8%
70's	14.8%
60's	34%
50's	30%
40's	11.3%
30's	1.7%
20's	0.7%

* means before any regrading.

All stats & numbers above are before any regrading.